Disjoint directed and undirected paths cycles and trees in directed graphs, Shandong Course 2026

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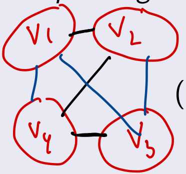
Edge-disjoint spanning trees in graphs

Theorem 1 (Tutte 1961)

An undirected graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if

$$\sum_{1 \le i < j \le p} |(V_i, V_j)| \ge k(p-1), \tag{1}$$



holds for every partition $V_1, V_2, \ldots, V_p$ of V. Here $|(V_i, V_j)|$ denotes the number of edges with one end in V_i and the other in V_j.

Deciding whether a graph has k edge-disjoint trees and finding the desired trees if they exist can be done in polynomial time via matroid partition algorithms.

The characterization also holds for multigraphs.

Arc-disjoint out-branchings in digraphs

An **out-branching** rooted at s in a digraph $D = (V, A)$ is a connected subdigraph B_s^+ of D where each vertex distinct from s has in-degree 1 and s has in-degree 0. I.e it is an orientation of a spanning tree of $UG(D)$ such that s can reach every other vertex by a directed path.

Theorem 2 (Edmonds 1973)

A digraph $D = (V, A)$ has k arc-disjoint out-branchings rooted at $s \in V$ if and only if

$$d^-(X) \geq k \quad \forall \quad \emptyset \neq X \subseteq V - s. \tag{2}$$

Deciding whether a digraph has k arc-disjoint out-branchings rooted at a given vertex s and finding the desired branchings if they exist can be done in polynomial time via flows

Thomassé's question

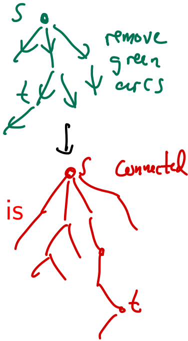
The motivation for this kind of research came partly from the following question due to Thomassé.

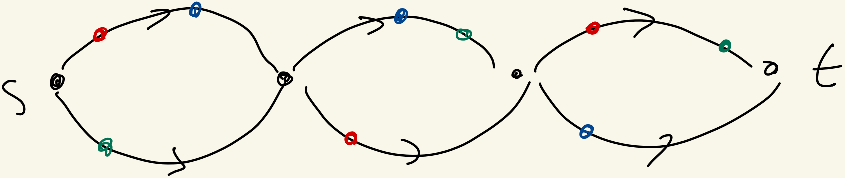
Problem 3 (Thomassé 2007)

Find a good characterization of directed graphs having two disjoint directed spanning trees such that one of the spanning trees is an out-branching rooted at a given vertex.

Equivalently: Characterize those digraphs which have an out-branching B_s^+ rooted at a given vertex s such that $UG(D - A(B_s^+))$ is connected.

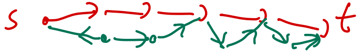
Zirui proved in his presentation that Thomassé's question is NP-complete!




s
t

(arc-)disjoint (s,t)-paths in (di)graphs

- By Menger's theorem, a (di)graph H has two edge-disjoint (arc-disjoint) (s,t)-paths if and only if there there is no edge (arc) whose removal destroys all (s,t)-paths.
- Easy to find the desired paths in linear time if they exist.
- What about the mixed version: Given a digraph D and vertices s,t. Does $UG(D)$ contain two edge-disjoint (s,t)-paths P,Q such that P is also a directed (s,t)-path in D?



- If a digraph D contains a branching B_s^+ such that $UG(D-A(B_s^+))$ is connected, then for every other vertex t, $UG(D)$ contains two edge-disjoint st-paths P,Q so that P is a directed (s,t)-path in D.

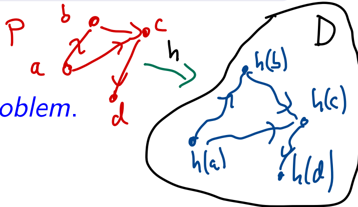
Linkage problems

FORTUNE, HOPCROFT, and WYLLIE proved that most directed linkage problems are $\mathcal{NP}$-complete. To be more precise let us consider the following linkage problem. We fix a digraph P.

Problem 4 (Directed linkage with demand digraph P)

Given a digraph D and an injection $h : V(P) \to V(D)$, decide if h extends to an injection on $V(P) \cup A(P)$ such that, for every arc $a = st$ of P, $h(a)$ is an $(h(s), h(t))$-path in D if $s \neq t$ and a cycle in D containing $h(s)$ if $s = t$, and, for each $b \in A(P) - \{a\}$, $V(h(a)) \cap V(h(b)) \subseteq \{h(s), h(t)\}$.

FORTUNE et al. call this the
Fixed directed subgraph homeomorphism problem.



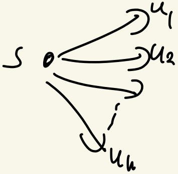
Theorem 5 (Fortune, Hopcroft and Wyllie 1980)

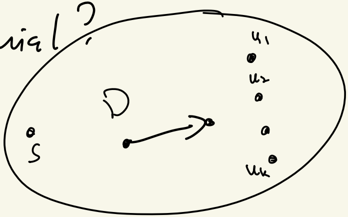
Assuming $\mathcal{P} \neq \mathcal{NP}$, the linkage problem with demand digraph P is polynomially solvable precisely when all arcs of P have the same head or they all have the same tail.

It is an easy consequence of Graph Minors XIII that the undirected analogue is polynomially solvable for any fixed demand graph.

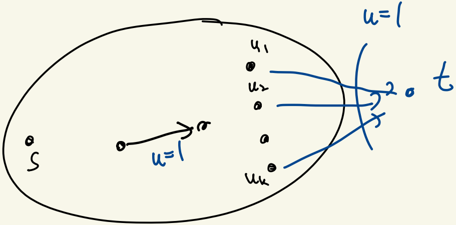
Theorem 6 (Fortune, Hopcroft and Wyllie, 1980)

For every demand digraph P, there is a polynomial time algorithm to decide if a given directed acyclic digraph D and an injection $h : V(P) \rightarrow V(D)$ admits an extension of h to a homeomorphism from P to D.

Why is  polynomial?



N_D



N_D has an (s,t)-flow of value k

$\Updownarrow$ paths exist in D

Let us fix a mixed graph $P = (V', E' \cup A')$ with edges E' and arcs A', and consider a digraph D. We call an injection h on $V' \cup A' \cup E'$ a *mixed homeomorphism from P to D*, if

(H1) for every vertex x of P, $h(x)$ is a vertex of D,

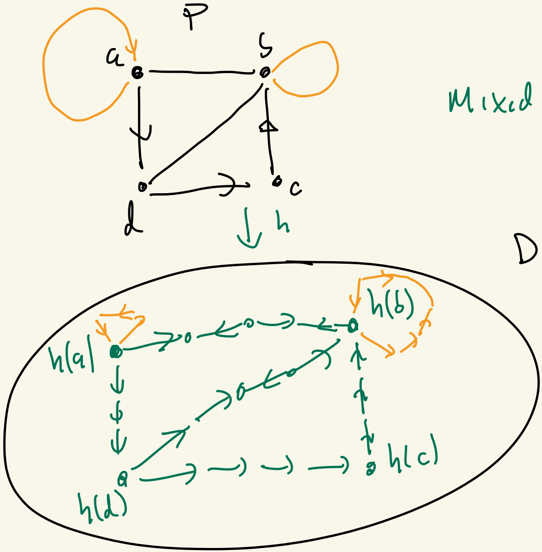
(H2) for every arc $a = st$ of P, $h(a)$ is an $(h(s), h(t))$-path in D if $s \neq t$ and a cycle in D containing $h(s)$ if $s = t$,

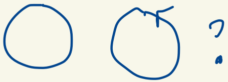
(H3) for every edge $e = st$ of P, $h(e)$ is an $h(s)h(t)$-path in $UG(D)$ if $s \neq t$ and a cycle in $UG(D)$ containing $h(s)$ if $s = t$, and

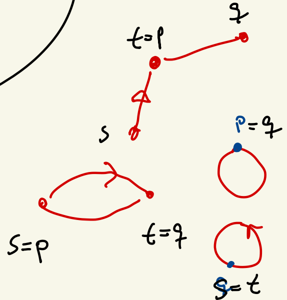
(H4) for distinct $x, y \in A' \cup E'$, $V(h(x)) \cap V(h(y)) \subseteq h(V(x))$,

where $V(x)$ denotes the set of endvertices of an arc or edge from P

Mixed linkage







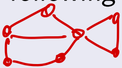
Problem 7 (Directed linkage with mixed demand graph P)

Given a directed graph D and an injection $h : V(P) \to V(D)$, decide whether h extends to a mixed homeomorphism from P to D.

Theorem 8

The directed linkage problem with mixed demand graph P is polynomially solvable in the following cases:

(a) *P has no arcs, or*



polynomial by Robertson + Seymour

(b) *P has no edges and there is some vertex s in $V(P)$ that is either the head of all arcs in P or the tail of all arcs in P.*

The problem is $\mathcal{NP}$-complete for all other mixed demand graphs P.

Corollary 9

The following problems are all $\mathcal{NP}$-complete for digraphs. Decide whether for a given input digraph D and vertices $s \neq t, p \neq q$, there exist

(a) *a cycle B in D containing s and a cycle C in $UG(D)$ containing p with $V(B) \cap V(C) \subseteq \{s\} \cap \{p\}$;*

(b) *an (s,t)-path P in D and a cycle C in $UG(D)$ containing p with $V(P) \cap V(C) \subseteq \{s,t\} \cap \{p\}$;*

(c) *a cycle B in D containing s and a pq-path Q in $UG(D)$ with $V(B) \cap V(Q) \subseteq \{s\} \cap \{p,q\}$;*

(d) *an (s,t)-path P in D and a pq-path Q in $UG(D)$ with $V(P) \cap V(Q) \subseteq \{s,t\} \cap \{p,q\}$.* □

Note that, by Corollary 9(d), it is already $\mathcal{NP}$-complete to decide whether the underlying graph of a given digraph D contains two internally disjoint (s, t)-paths P_1, P_2 so that P_1 is also a path in D.

Theorem 10

The directed linkage problem with mixed demand graph P is polynomially solvable for acyclic digraphs in the following cases:

(a) *P has no arcs, or* by Seymour-Robertson

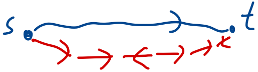
(b) *P has no edges, or* by Fortune et al

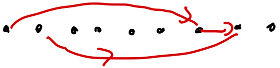
(c) *P contains a directed cycle.* trivial

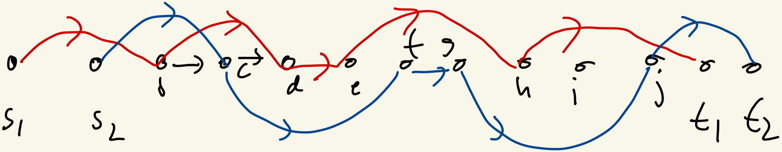
The problem is $\mathcal{NP}$-complete for all other mixed demand graphs.

Corollary 11

It is NP-complete to decide for a given acyclic digraph D and vertices s, t whether $UG(D)$ contains contains two edge-disjoint st-paths P, Q so that P is a directed (s, t)-path in D.







$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \to \begin{pmatrix} b \\ s_2 \end{pmatrix} \to \begin{pmatrix} b \\ c \end{pmatrix} \to \begin{pmatrix} d \\ c \end{pmatrix} \to \begin{pmatrix} e \\ c \end{pmatrix} \cdots \quad \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Disjoint (un)directed cycles in a digraph

- Lovász (1965) characterized those undirected graphs that do not have two disjoint cycles. The characterization leads to a polynomial algorithm to decide the existence of such cycles.
- A digraph is **inter-cyclic** if it does not contain two disjoint cycles.
- McQuaig characterized inter-cyclic digraphs with minimum in- and out-degree at least 2.
- His characterization leads to a polynomial algorithm for deciding whether a given digraph is inter-cyclic.

Problem 12

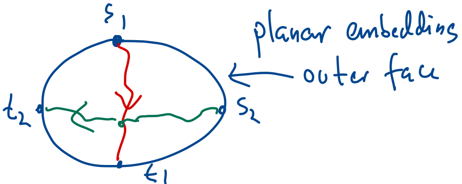
What is the complexity of deciding for a given digraph D whether $UG(D)$ contains two disjoint cycles B, C such that B is also a cycle in D? Solution will be provided later in the talk

Theorem 13

There exists a polynomial algorithm for deciding whether the underlying graph of a given strongly connected digraph D contains two disjoint cycles B, C so that B is also a cycle of D.

Seems non-trivial to prove. Our proof uses

- McCuaig's characterization of inter-cyclic digraphs,
- Thomassen's solution of the 2-path-problem for acyclic digraphs and
- a non-trivial algorithm for the case of cycle transversal number one.



The mixed disjoint cycles problem

MIXED-DISJOINT-CYCLES
Input: A digraph D
Question: Decide if there exists cycles B in D and C in $UG(D)$ such that $V(B) \cap V(C) = \emptyset$.

Theorem. (Bang-Jensen & Kriesell 2009)
MIXED-DISJOINT-CYCLES restricted to strongly connected digraphs D is in $\mathcal{P}$, and the desired cycles B, C can be found in polynomial time if they exist.

When do there exist disjoint cycles B in D and C in $UG(D)$?

Obv. necessary:	two disjoint cycles in $UG(D)$.
Obv. sufficient:	two disjoint cycles in D.

Concentrate on digraphs $\underbrace{\text{without two disjoint cycles}}_{=:\ \text{intercyclic}}$.

Suppose D is strongly connected.
Distinguish cases according to the cycle transversal number

$$\tau(D) := \min\{|T| : T \subseteq V(D),\ D - T \text{ acyclic}\}.$$

$\tau(D)$ can be determined in polynomial time (McCuaig 1993).

$\tau(D)$	Method to answer
0	"No" (as D is acyclic)
1	Decision Algorithm
2	Characterization (partly topological argument)
3	"Yes" (using a result of McCuaig)
≥ 4	"Yes" (as D is not intercyclic (McCuaig 1993))

Outline of our decision algorithm
for a strongly connected input digraph D_0 with $\tau(D_0) = 1$.

- Find a cycle transversal $\{a\}$ in D_0.
- Split a, i.e. replace a by a^+, a^-; for each edge from x to y, replace x by a^+ if $x = a$ and y by a^- if $y = a$.
- The result D is acyclic with source a^+ and sink a^-.
- Find a largest system $\mathcal{P}$ of openly disjoint (a^+, a^-)-paths.
- Study the $\bigcup \mathcal{P}$-bridges. Easy if $|\mathcal{P}| \neq 2$; reduce or decide.
- For $\mathcal{P} = \{P, Q\}$ reduce to the case that $V(D) = V(P) \cup V(Q)$; reduce further or decide.

Cycle transversal number 2 I

D_0 strongly connected, cycle transversal number is 2.
First: D_0 simple, minimal in- and out-degree at least 2.

- If D_0 is 2-regular then the answer is "no" if and only if D_0 is ~~not~~ the square of an odd cycle (1-page argument).
- Find cycle transversal $\{x, y\}$, split into x^+, x^- & y^+, y^-.
- The result D is acyclic; sources: x^+, y^+, sinks: x^-, y^-.
- If D_0 is not intercyclic then the answer is "yes".
- Otherwise, find a plane embedding of D in the unit disc S where x^+, y^+, x^-, y^- map to $(-1, 0), (0, -1), (1, 0), (0, 1)$ and these are the only points on ∂S.
  (McCuaig 1993, Metzlar 1989, Thomassen 1983/1985)
- Find cycles B, C as required ("yes"),
  unless D_0 is a diwheel ("no").

Cycle transversal number 2 II

D_1 strongly connected, cycle transversal number is 2.

- If vertex x has only one outneighbor y: Contract xy.
- If vertex x has only one in-neighbor y: Contract yx.
- Repeat the first two steps as long as possible.
  The underlying simple digraph D_0 of the result
  has minimum in- and out-degree ≥ 2.
- Decide for D_0.
  If the answer for D_0 is "yes" then the answer for D_1 is "yes".
- Otherwise analyze
  - the subdigraphs formed by the contracted edges, and
  - the edges connecting them.

  (Reorganized representation of the input D_1.)
  Difficult, many different "yes"- and "no"-instances.

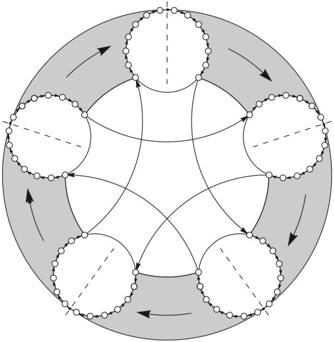


Figure 5. A typical vault. The five central arcs must have multiplicity 1 and are the only arcs from P_i to P_{i+2}.

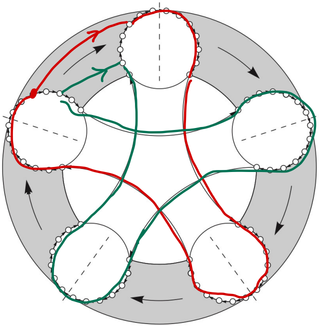
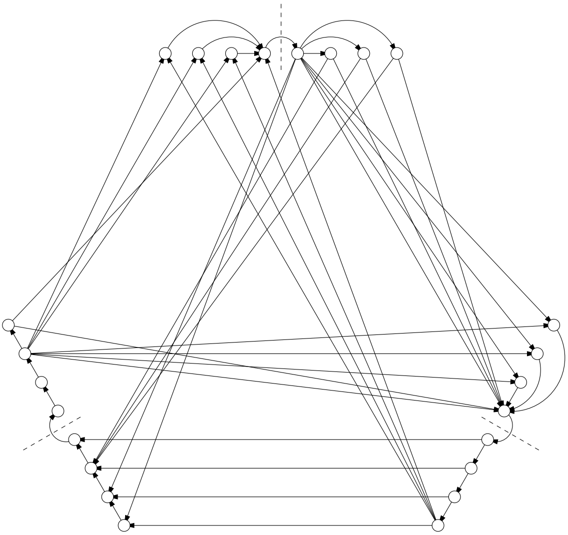


Figure 5. A typical vault. The five central arcs must have multiplicity 1 and are the only arcs from P_i to P_{i+2}.

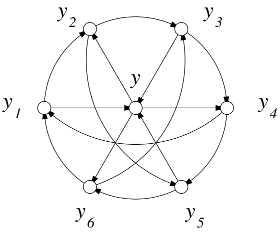
A trivault

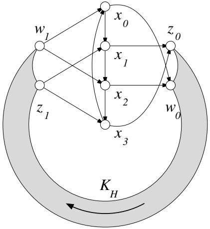


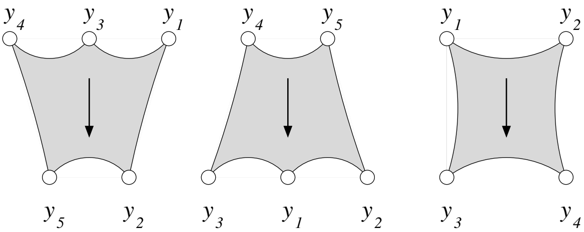
Typical example; D_0 is the square of a triangle.

Cycle transversal number ≥ 3

Structure theorem for intercyclic digraphs D (McCuaig 1993) implies "yes" for $\tau(D) \geq 3$.







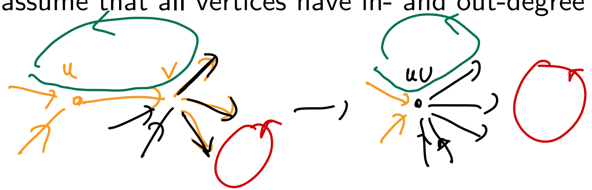
Why so difficult?

In the mixed version we can not, like in the directed case, employ three important concepts:

(i) Symmetry of the objects we are looking for,

(ii) strongly connectedness — the general directed version reduces immediately to this case, whereas here we needed to add it as a condition to the input digraph —, and, finally,

(iii) reduction by contracting an arc which is either the unique out-arc at their tail or the unique in-arc at their tail — in the directed case, this does not change the answer, so that we can immediately assume that all vertices have in- and out-degree at least 2.



The case when D is not strongly connected

Theorem 14 (Bang-Jensen, Kriesell, Madaloni, Simonsen, 2013)

The MIXED-DISJOINT-CYCLES *is polynomially solvable in the following cases*

- *The input is strongly connected (Theorem 13)*
- *The input is not strong and every cycle transversal has size at least 2.*
- *The input is not strong and the number of cycle transversals of size 1 is bounded (by some constant).*
- *If the input has no directed cycle*

In the remaining case when there can be an unbounded number of cycle transversals of size 1, the problem is NP-complete.

RESTRICTED-CYCLE-IN-BIPARTITE-GRAPHS

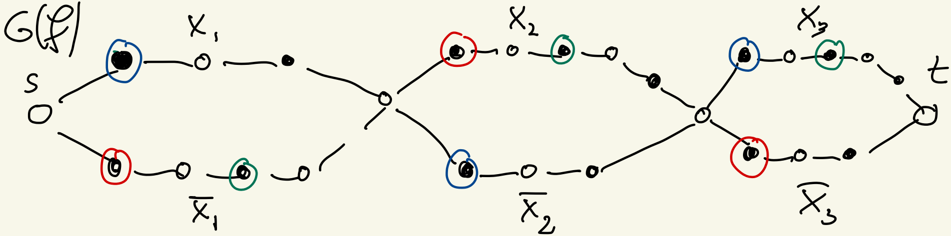
Input: A bipartite graph $G = (U, V, E)$ with colour classes U and V and a partitioning $V_1, V_2, \ldots, V_k$ be of V into disjoint non-empty sets.

Question: Decide if there exists a cycle C in G which avoids at least one vertex from each V_i.

Lemma 15

RESTRICTED-CYCLE-IN-BIPARTITE-GRAPHS *is $\mathcal{NP}$-complete.*

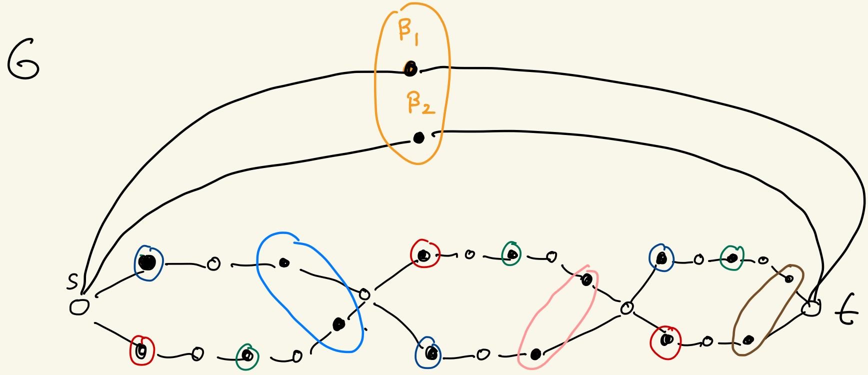
$$f = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$$



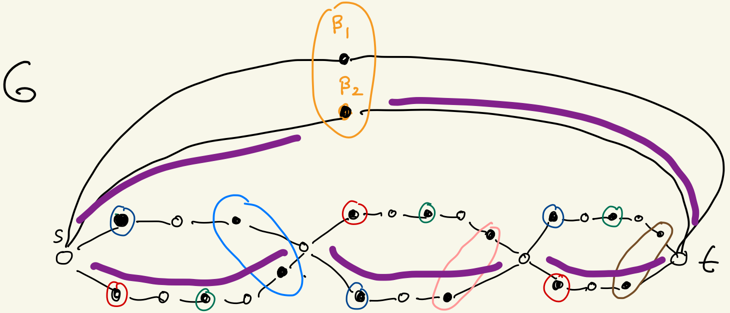
G(f) has an (s,t)-path avoiding at least one vertex from each clause triple

⇕

f is satisfiable



The colored sets form a partition of the black vertices, that is, V into sets of size 3 (clauses) and 2



Every cycle that contains at most one vertex from each of the V_j's of size 2 is of the form

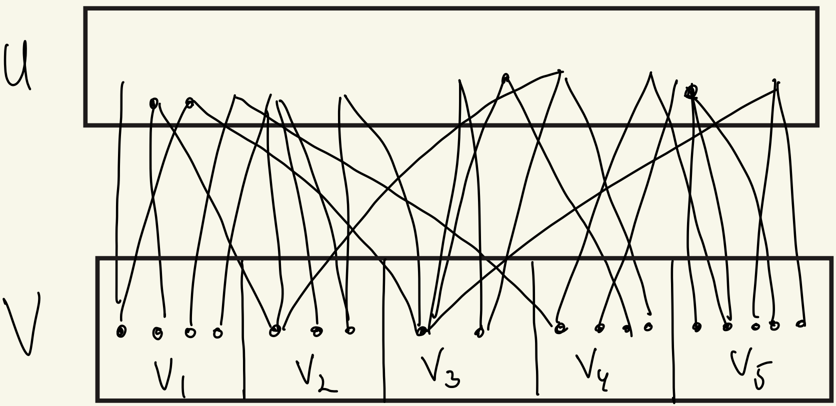
$sPt\beta_i s$ when P is an (s,t)-path in $G(f)$

<u>Conclusion:</u> G has a cycle avoiding at least one vertex in each $V_i \Leftarrow f$ is satisfiable

We describe a polynomial reduction from RESTRICTED-CYCLE-IN-BIPARTITE-GRAPHS to the MIXED-DISJOINT-CYCLES problem restricted to the case of dicycle transversal number 1 and an unbounded number of transversal vertices.

Let $H = (U, V, E)$ be a bipartite graph with colour classes U, V where $U = \{b_1, ..., b_r\}$, and $V = V_1 \cup V_2 \cup \ldots \cup V_k$ with $V_i = \{p_{i,1}, ..., p_{i,\ell_i}\}$, $\ell_i > 0$ for $i \in \{1, 2, \ldots, k\}$, and $V_i \cap V_j = \emptyset$ if $i \neq j$.

We construct a directed graph D from H in the following way: Direct all edges of H from V to U. Add vertices $v_0, \ldots, v_{k+1}$. Add arcs $v_{i-1}p_{i,j}$ and $p_{i,j}v_i$ for all $i \in \{1, \ldots, k\}$,$j \in \{1, \ldots, l_i\}$. Finally add the arc $v_k v_0$.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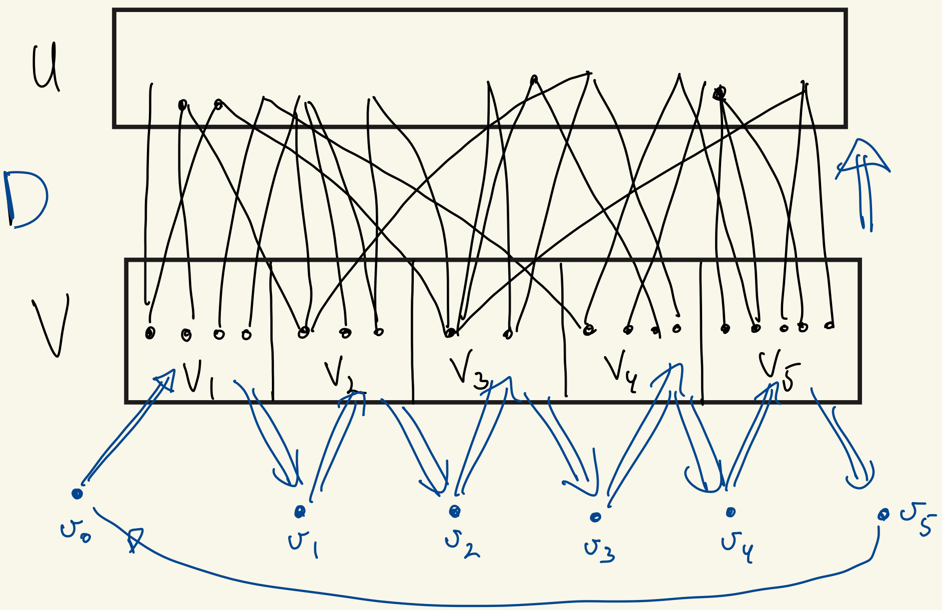

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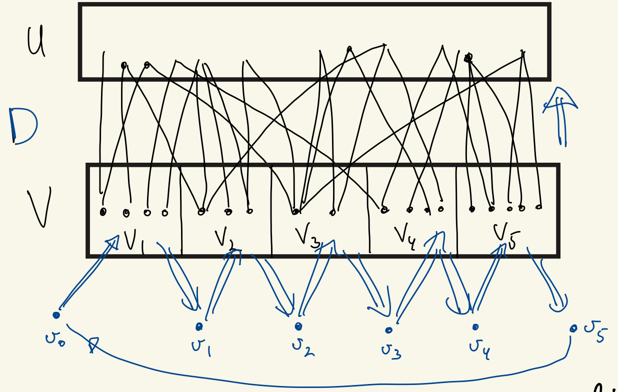

D has a directed cycle C which is disjoint from some 'undirected' cycle B

⇕

G has a cycle that avoids at least one vertex from each V_i

We claim that D contains a dicycle B and a cycle C of $UG(D)$ which are disjoint if and only if there is a cycle in H avoiding at least one vertex of V_i for each $i \in \{1, 2, \dots, k\}$. First suppose there is a cycle in H avoiding the vertex p_{i,a_i} of V_i for each $i \in \{1, 2, \dots, k\}$. Then, by the construction of D, the same cycle will be a cycle in $UG(D)$. The cycle $v_0 p_{1,a_1} v_1 p_{2,a_2} \dots v_{k-1} p_{k,a_k} v_k v_0$ is vertex disjoint from this undirected cycle, and we are done. Now suppose there is an undirected cycle C disjoint from some dicycle in D. Note that every dicycle in D is formed by the arc $v_k v_0$ and some (v_0, v_k)-path. The path is of the form $v_0 p_{1,a_1} v_1 \dots v_{k-1} p_{k,a_k} v_k$. Hence C does not contain any of the vertices $v_0, v_1, \dots, v_k$ and hence uses only $p_{i,j}$ or b_ℓ vertices and always alternates between them. Therefore C has a corresponding cycle in H, and this one avoids at least the vertex p_{i,a_i} from of the set V_i for each $i \in \{1, \dots, k\}$. □

Arc-disjoint cycle and dicycle

The arc-disjoint analogue of MIXED-DISJOINT-CYCLES seems at first glance simpler, but turns out to be NP-complete when the digraphs in question may have arbitrarily many transversal arcs (an arc belonging to every dicycle of D).

The algorithms for the polynomial cases use, among other things, the algorithms for the polynomial cases of MIXED-DISJOINT-CYCLES. The proof of the NP-completeness result is more involved than in the case of vertex-disjoint cycles.

Theorem 16 (Bang-Jensen, Kriesell, Maddaloni, Simonsen, 2014)

The problem of deciding whether a given input digraph D contains a dicycle B and a cycle C in $UG(D)$ such that $A(B) \cap A(C) = \emptyset$ is polynomially solvable for the class of strong digraphs and for every class of digraphs with a constantly bounded number of transversal arcs. In the remaining case, when the digraphs may have arbitrarily many transversal arcs, the problem is NP-complete.

Open problem

What is the complexity of deciding whether the underlying graph $UG(D)$ of a digraph D has arc-disjoint cycle factors $\mathcal{F}_1, \mathcal{F}_2$ so that in D all cycles in $\mathcal{F}_1$ are directed?

Note 1 $UG(D)$ has two arc-disjoint cycle factors (Petersen)
$\Updownarrow$
$UG(D)$ has a spanning 4-regular subgraph
(polynomial via matching techniques)

Note 2 D has two arc-disjoint cycle factors
$\Updownarrow$
D contains a spanning 2-regular subdigraph
(polynomial via flows)

- Every 6-connected graph is 2-linked [Seymour 1980, Thomassen 1980].
- For every natural number k there exists and k-strong digraph which is not 2-linked [Thomassen 1991].

Let us call digraph D **2-mixed-linkable** if, for every choice of vertices $s_1, s_2, t_1, t_2 \in V(D)$, the underlying graph of D contains a pair of internally disjoint paths P_1, P_2 such that P_1 is an $s_1 t_1$ path in the underlying graph of D and P_2 is a directed (s_2, t_2)-path.

Problem 17

Does there exist an integer N so that every N-strong digraph is 2-mixed-linkable?

Non-separating spanning subgraphs of digraphs

Conjecture 18 (Bang-Jensen and Yeo, 2002)

There exists an integer N such that every N-arc-strong digraph D contains arc-disjoint spanning strong subdigraphs D_1, D_2.

Conjecture 19

There exists an integer N such that every N-arc-strong digraph D contains a spanning strong subdigraph D' such that the underlying graph of $D - A(D')$ is (2-edge-)connected.

Theorem 20 (BJ+Yeo 09)

It is NP-complete to decide whether a 2-regular digraph D contains a spanning strong subdigraph D' such that the underlying graph of $D - A(D')$ is connected.

Theorem 21

It is $\mathcal{NP}$-complete to decide whether a given digraph has an (s,t)-path P such that $D - A(P)$ is connected for specified vertices s, t. □

Theorem 22

It is $\mathcal{NP}$-complete to decide for a given digraph D and distinct vertices vertex $s, t \in V(D)$ whether the underlying graph of D has an (s,t)-path Q such that $D - A(Q)$ has an out-branching B_s^+ rooted at s.

Theorem 23

It is $\mathcal{NP}$-complete to decide for a given strongly connected digraph D whether D contains a directed cycle C such that $UG(D - A(C))$ is connected.

Theorem 24

It is $\mathcal{NP}$-complete to decide for a given strongly connected digraph D whether $UG(D)$ contains a cycle C such that $D - A(C)$ is strongly connected.