A Survey on Advice and Randomization for the Knapsack Problem

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Online Problems and Advice Complexity
Introduction

Online Problem

- Sequence of requests
- Answer each request before the next one arrives
- Minimize cost / maximize gain
- Examples: Ski Rental, k-Server, Paging, Job Shop Scheduling

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If optimum is constant, $\alpha = 0$.

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- to be optimal?
- to achieve some competitive ratio?

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Motivation

- Theoretical interest: Deeper understanding of the problems
- "Essence" of the problem
- Bounds for randomization

The Model

Computation with Advice

Oracle with unlimited power:

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Analysis

- Solution: (oracle, algorithm)
- Correctness: the pair works correctly on all inputs
- Advice complexity: Maximal advice over all inputs of length $\leq n$

The Knapsack Problem

Introduction

Definition (KP)

Given a knapsack of weight capacity 1, n objects arrive in successive time steps where each object has

- a weight $w_i \leq 1$
- a value v_i

Maximize value of packed objects without exceeding capacity

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In its **simple** version (SKP), $w_i = v_i$, for every object i

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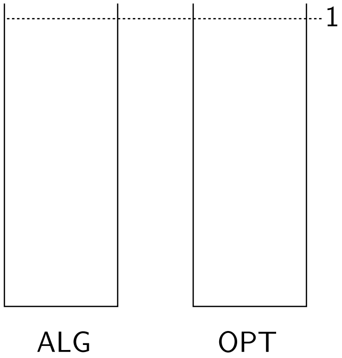

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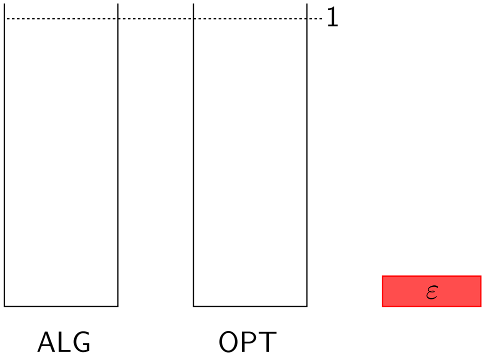

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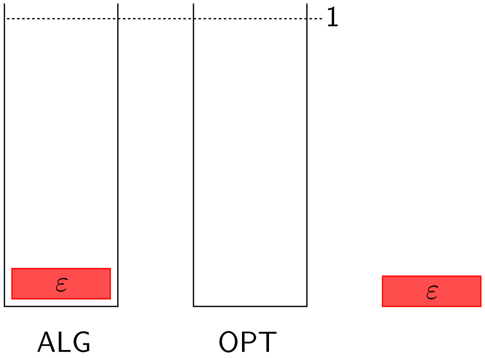

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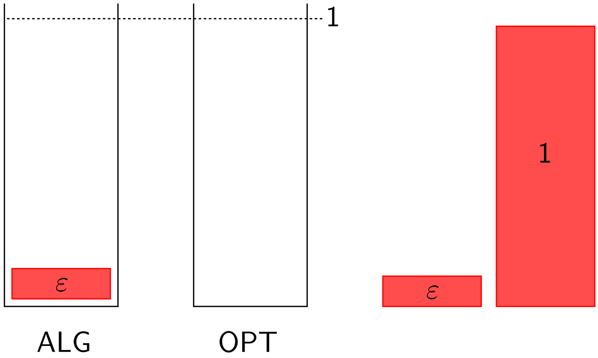

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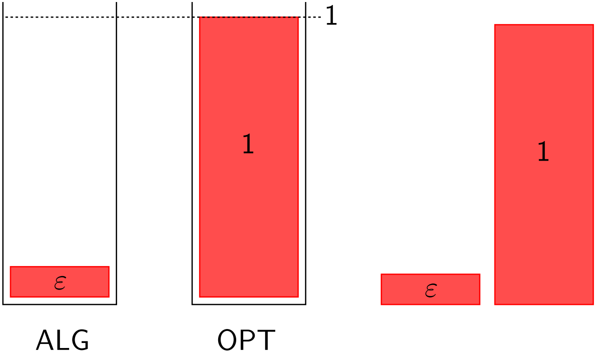

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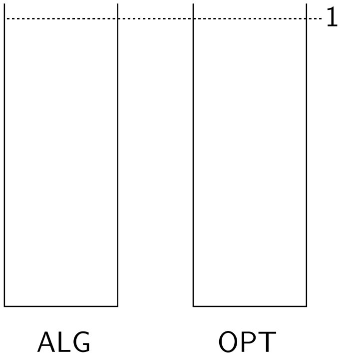

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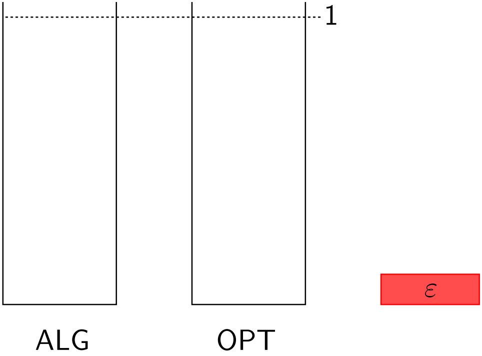

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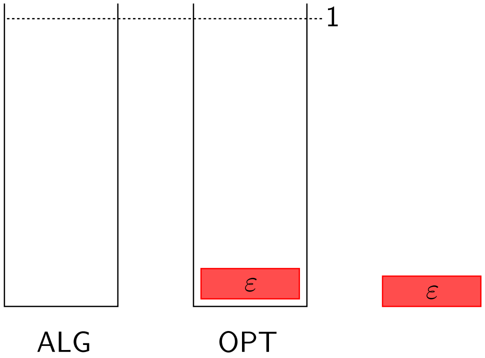

The Simple Knapsack Problem

Advice for Optimality

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- This bound is tight. In other words, ...

Theorem

Any online algorithm with advice for the SKP needs to use at least $n-1$ advice bits to be optimal.

The Simple Knapsack Problem

Small Advice

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- If sum of weights ≤ 1, greedy is optimal
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- Oracle writes 1 on tape iff one object has size $> 1/2$
- Consider ALG that reads one bit *b* of advice

If b = 0: `satisfy greedily`

If b = 1: `wait for object of size` $> 1/2$

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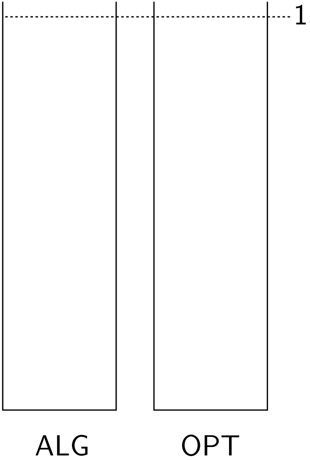

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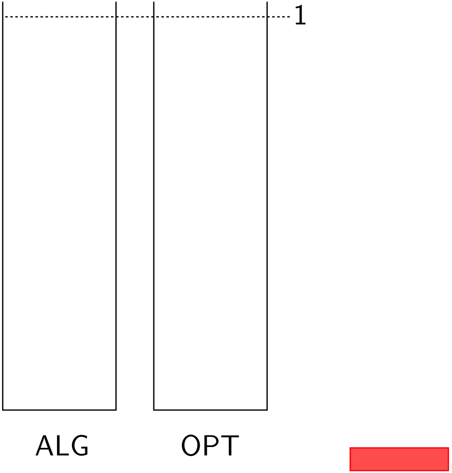

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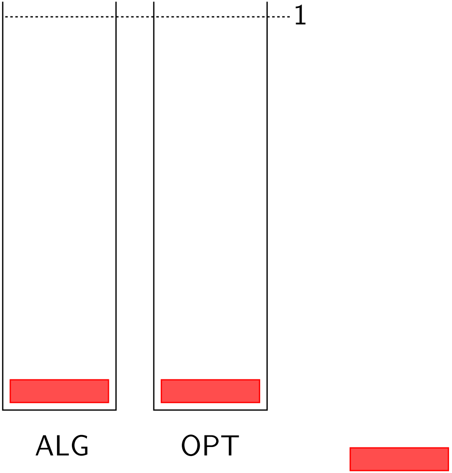

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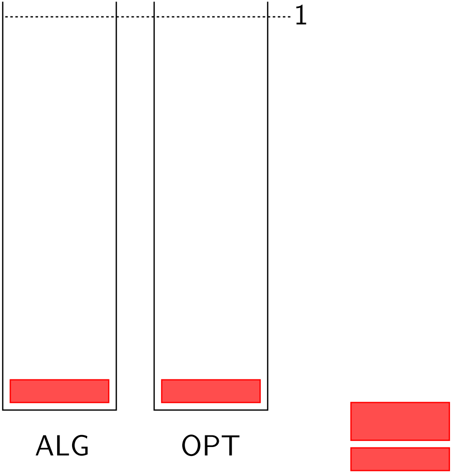

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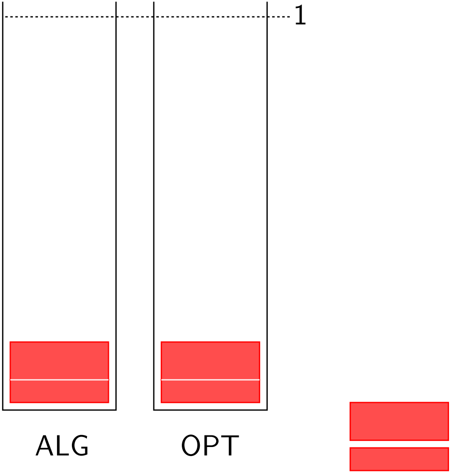

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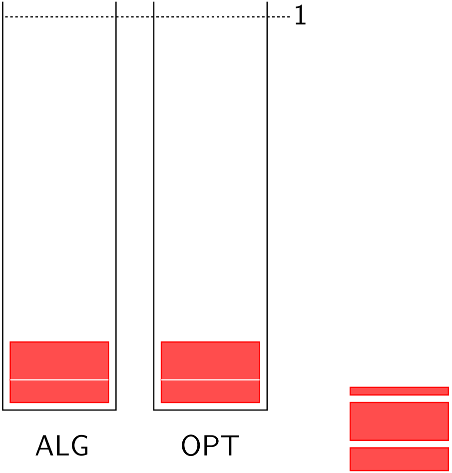

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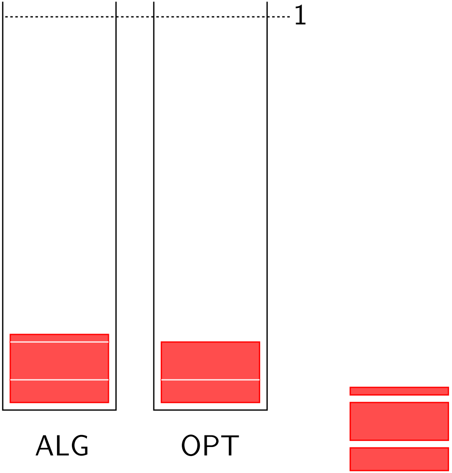

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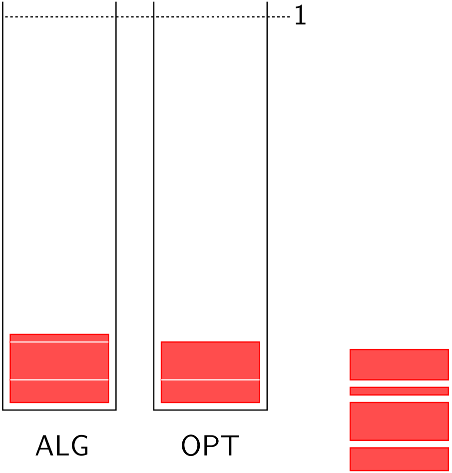

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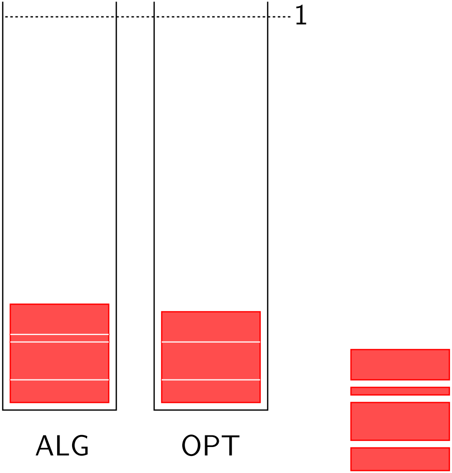

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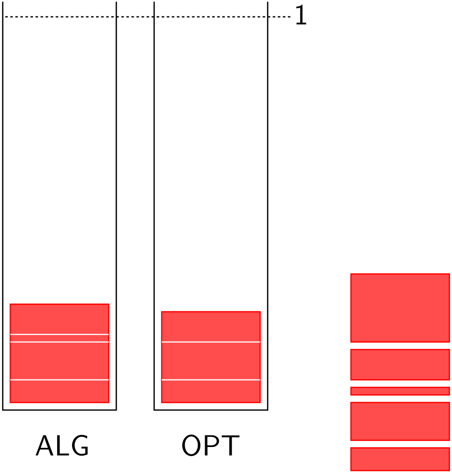

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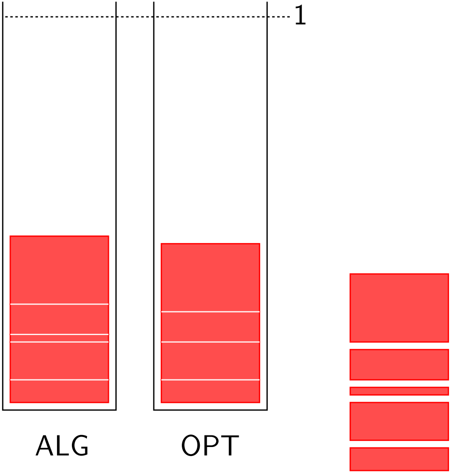

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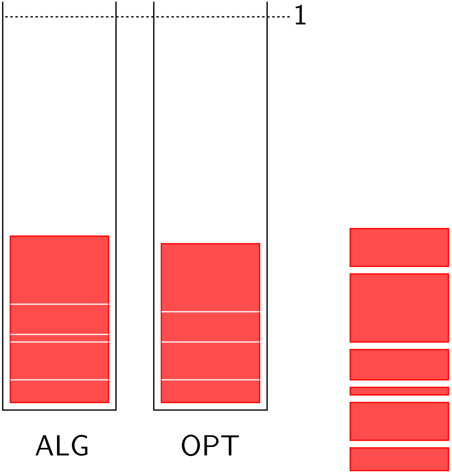

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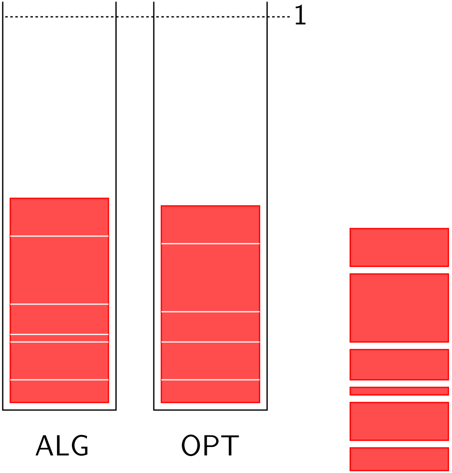

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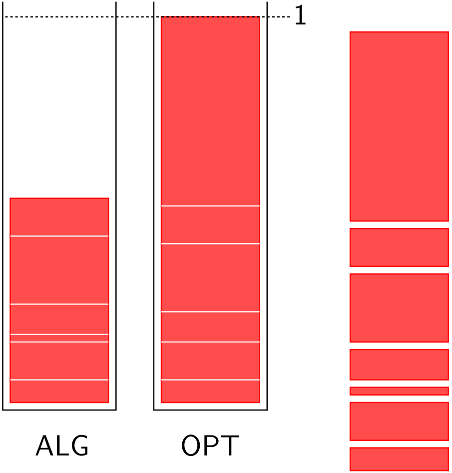

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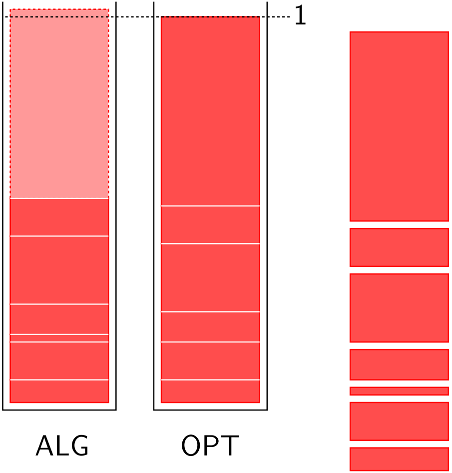

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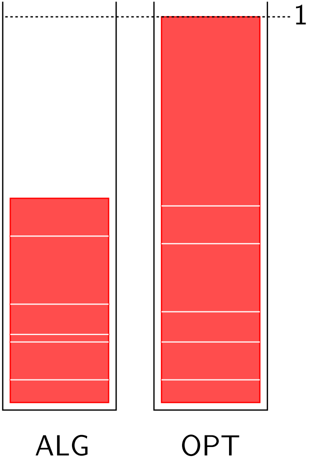

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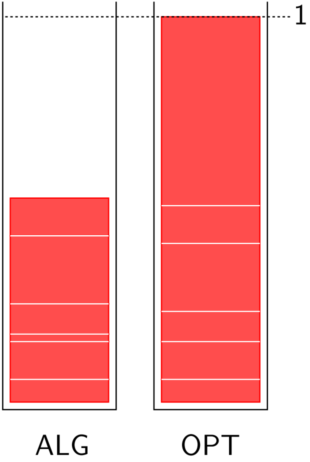

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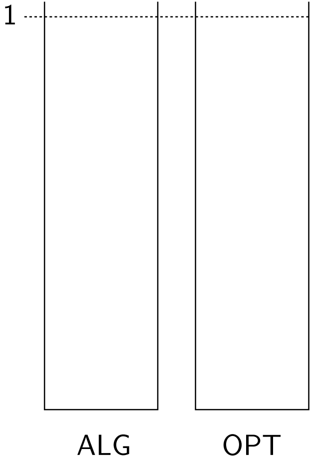

The Simple Knapsack Problem – Small Advice

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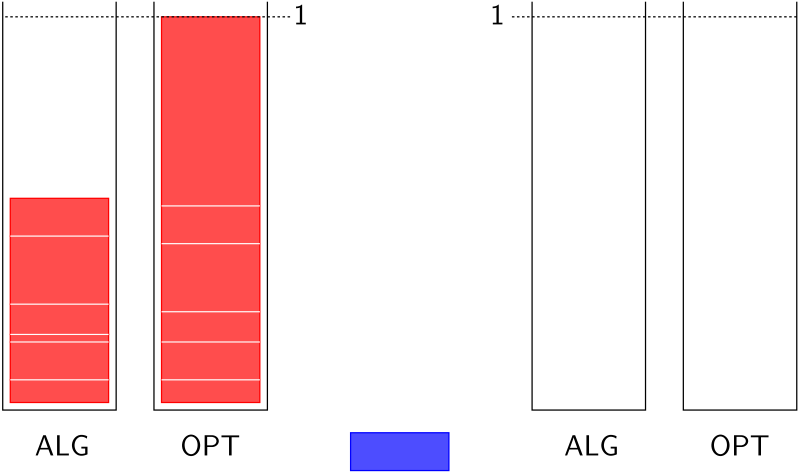

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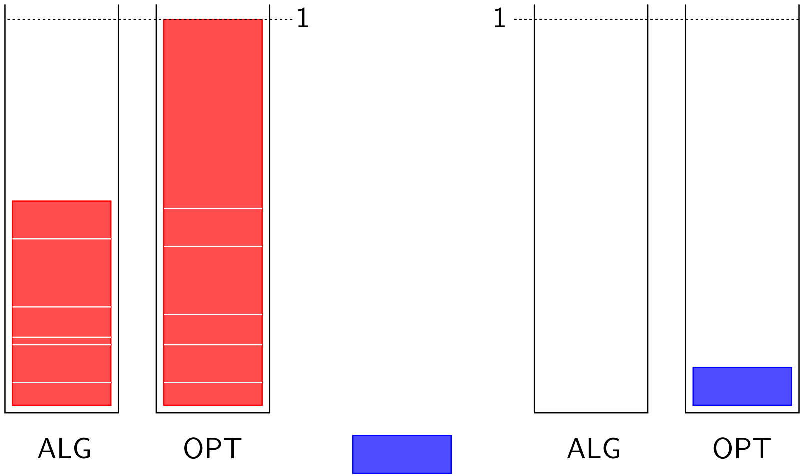

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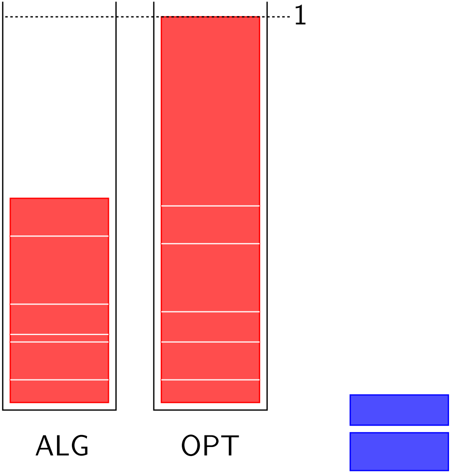

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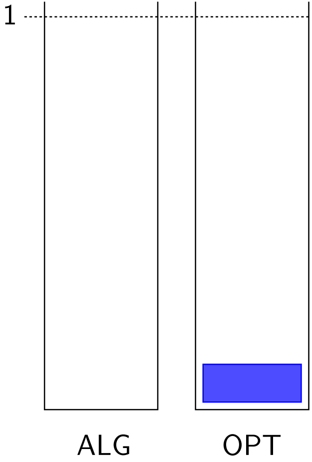

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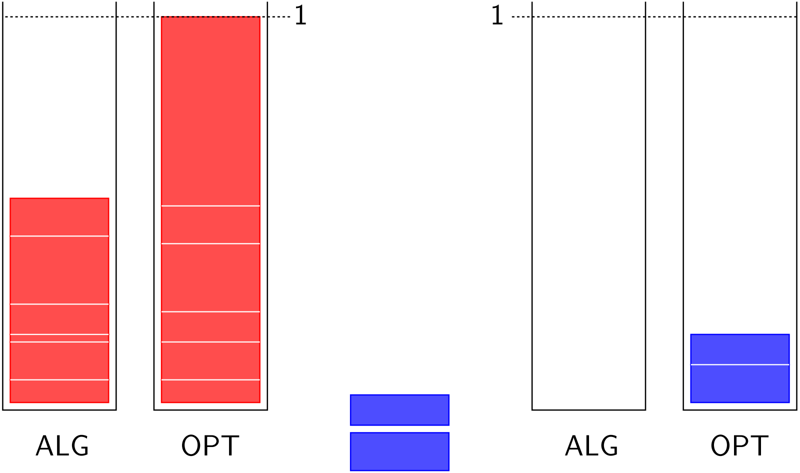

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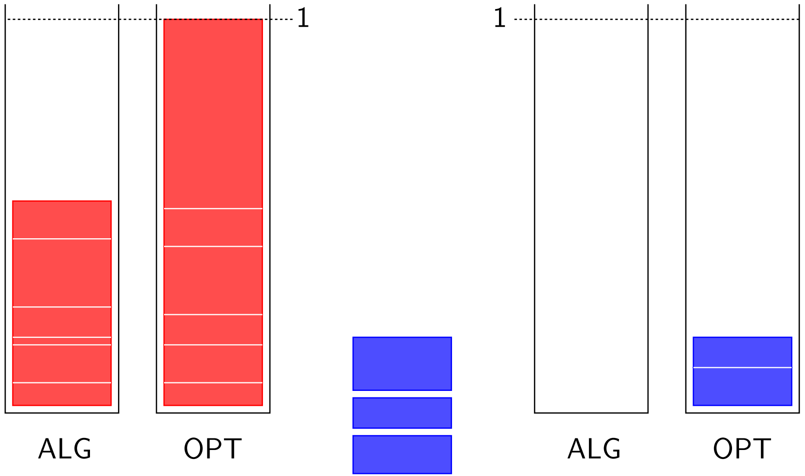

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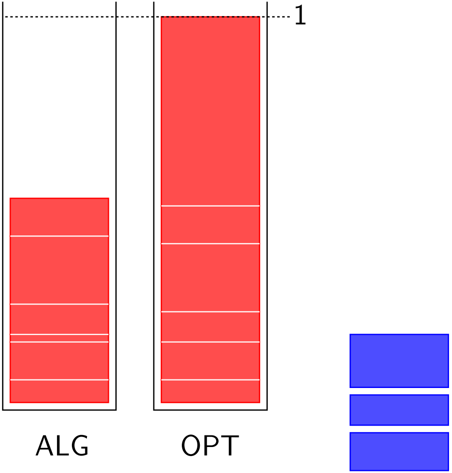

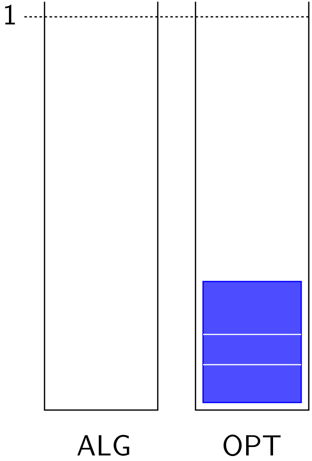

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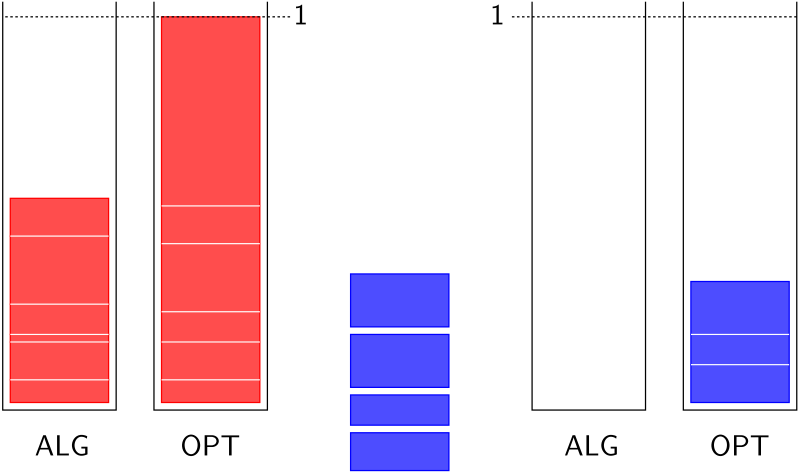

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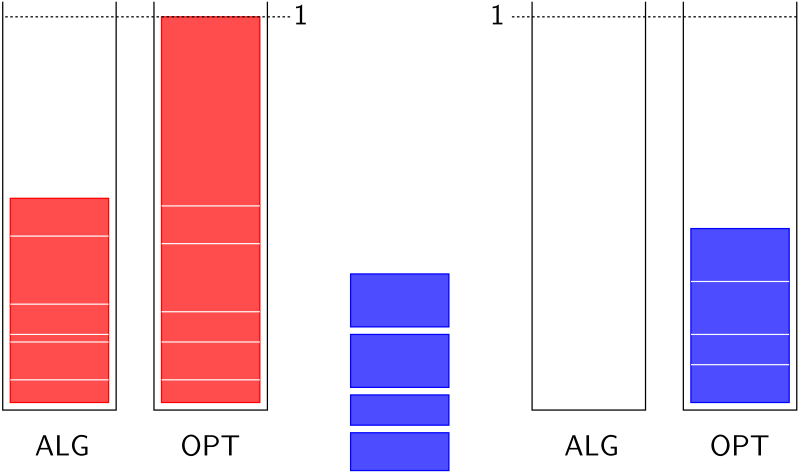

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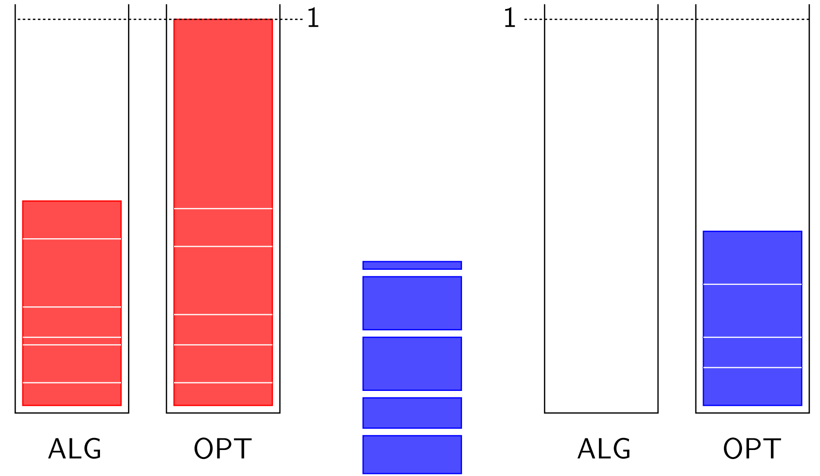

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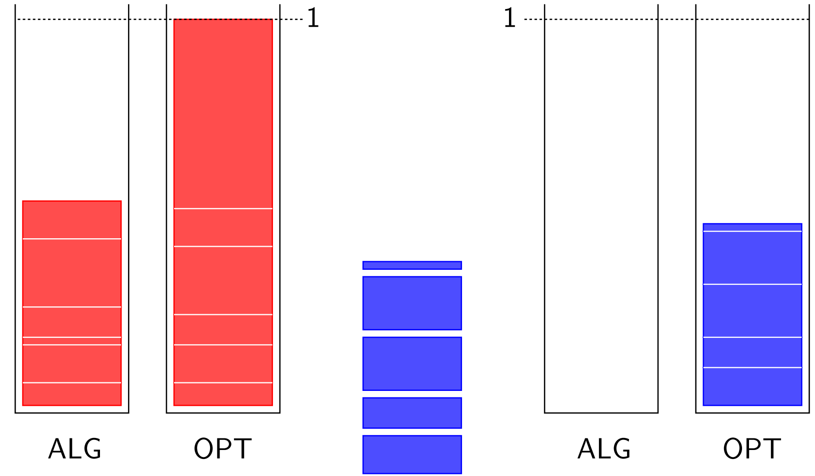

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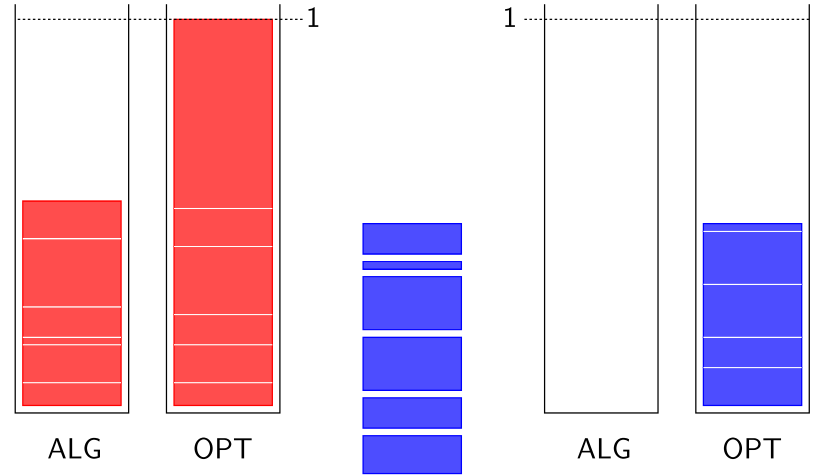

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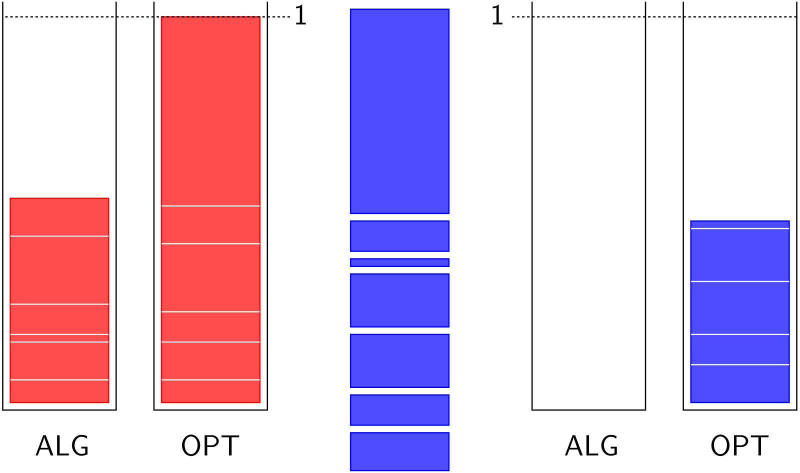

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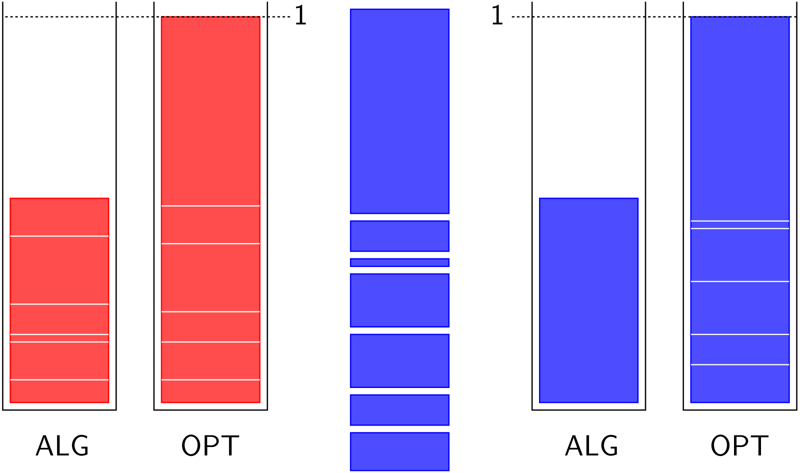

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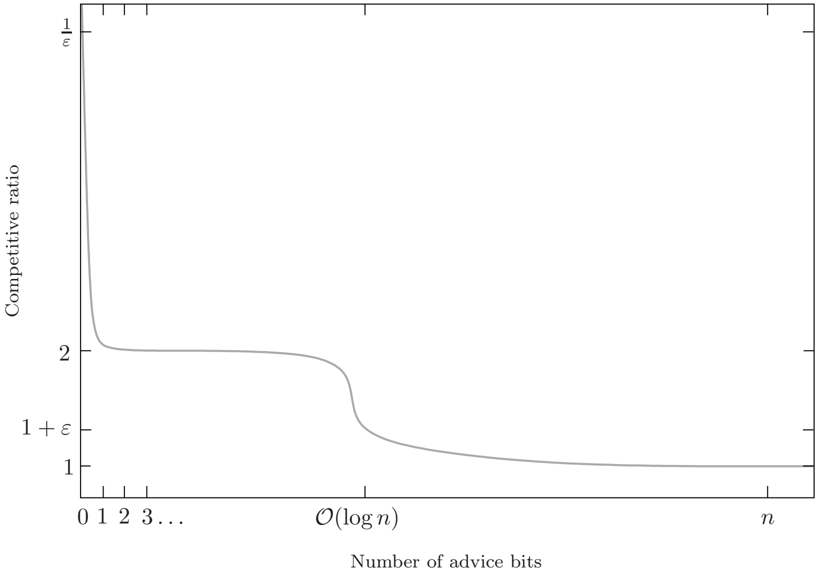
However, as we have seen before, an exponential jump has to be done to be **optimal** instead of only "**very well**"

The Simple Knapsack Problem – Small Advice



The General Knapsack Problem

Small Competitive Ratio

- No longer assume that weights and values are equal and ≤ 1

⇨ Weights are ≤ 1, values are possibly larger

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- Asymptotically equivalent to simple version
- ⇨ Constant of $\mathcal{O}$ notation is much worse

Advice and Randomization

Computation with Advice

- **Oracle** ⇨ Infinite **advice** tape ⇨ Algorithm
- Oracle: Knows whole input, unlimited computational power
- Advice tape prepared before the algorithm starts
- Advice complexity $b(n)$:
  Maximal number of bits read for inputs of length n

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Randomization

- **Random source** ⇨ Infinite **random** tape ⇨ Algorithm
- Random bit complexity $r(n)$:
  Maximal number of bits read for inputs of length n

Randomization and Advice

- $2^{b(n)}$ algorithms or $2^{r(n)}$ algorithms
- Advice may be seen as best random string for every instance

⇨ Lower bounds for advice carry over to randomization

⇨ Upper bounds for randomization carry over to advice

- Small advice may lead to **barely random algorithms**, e. g.,
  - Paging
  - Job Shop Scheduling
  - **Knapsack**

Advice and Randomization

Barely Random Algorithm for the Simple Knapsack Problem

Guess one bit and act as with advice

⇨ 4-competitive in expt. and this is tight

Guess one bit and act as with advice

- ⇨ 4-competitive in expt. and this is tight, but. . .

Theorem

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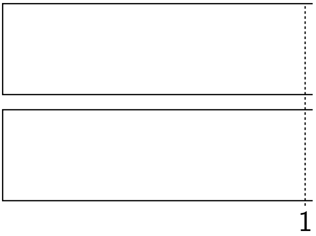

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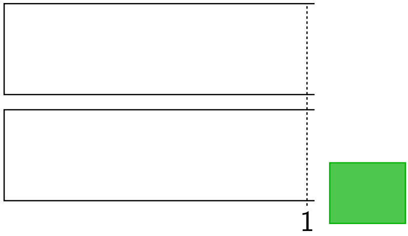

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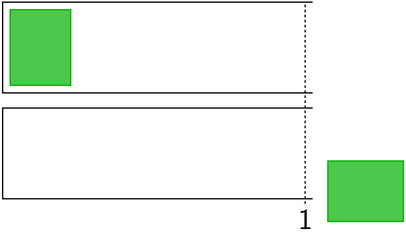

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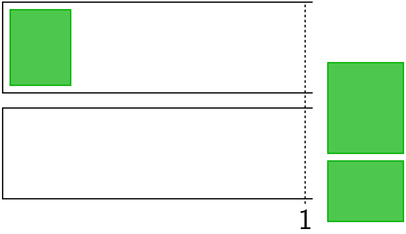

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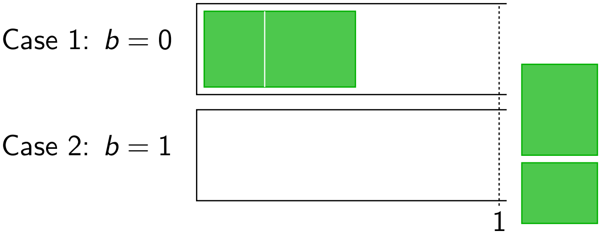

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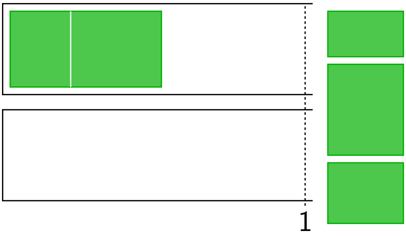

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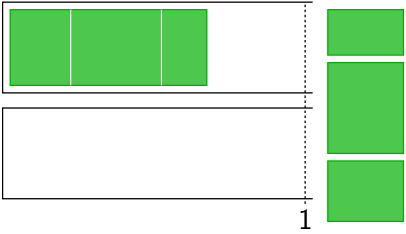

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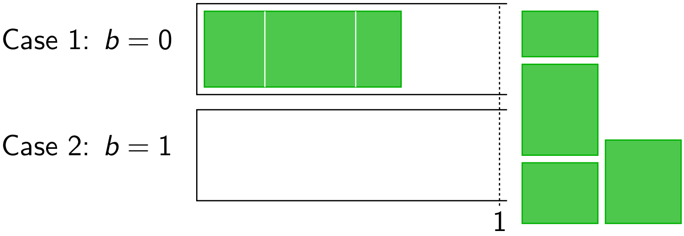

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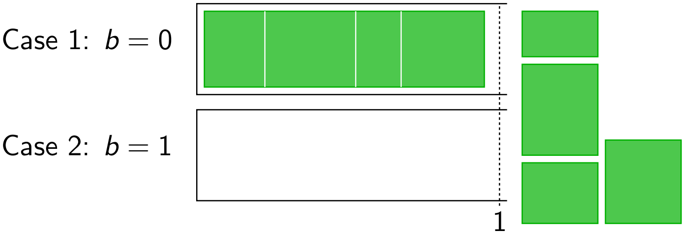

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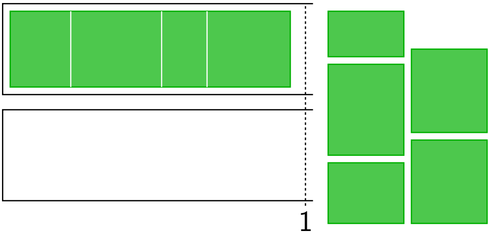

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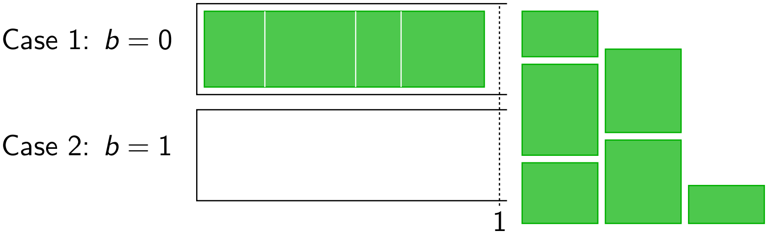

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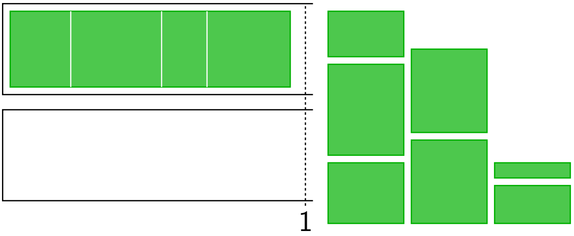

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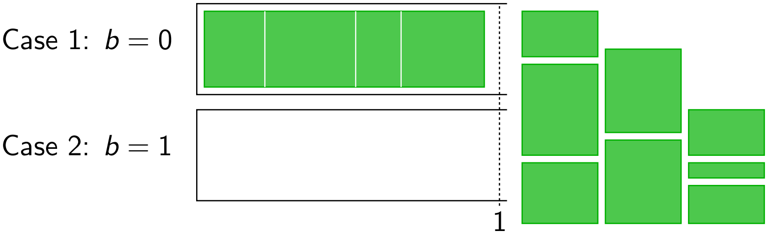

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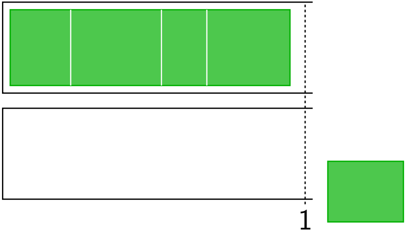

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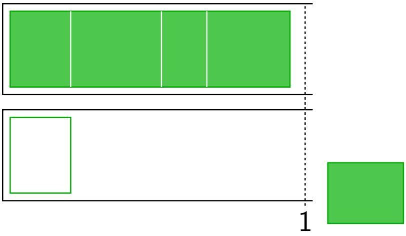

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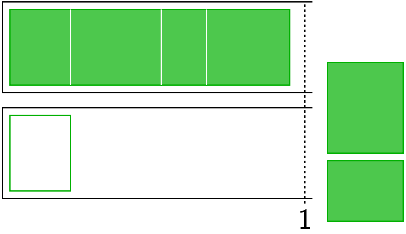

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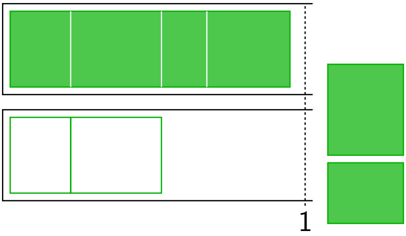

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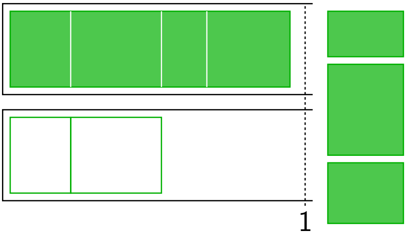

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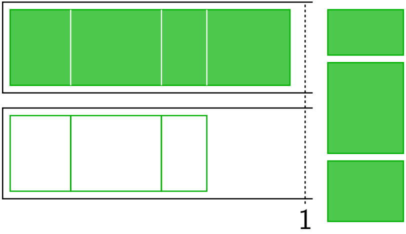

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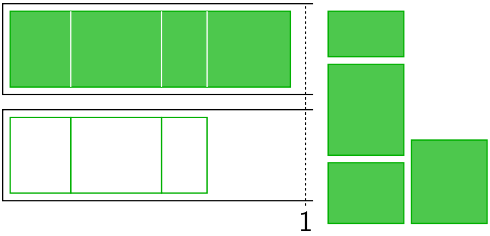

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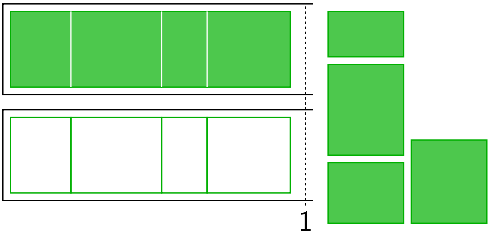

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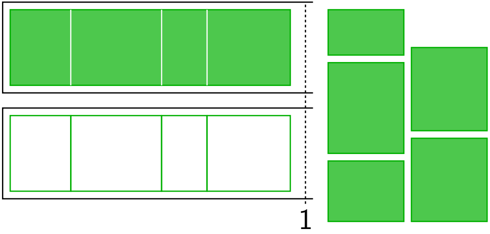

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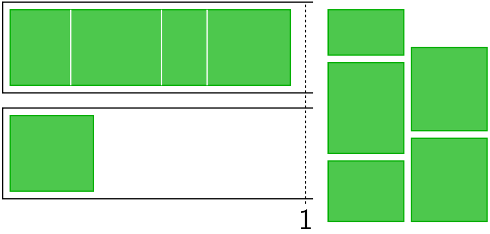

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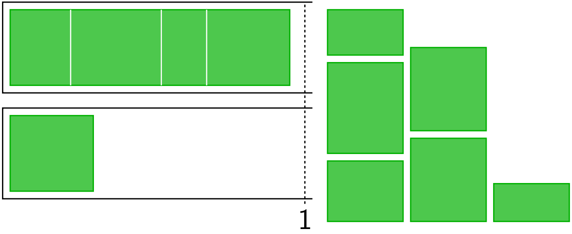

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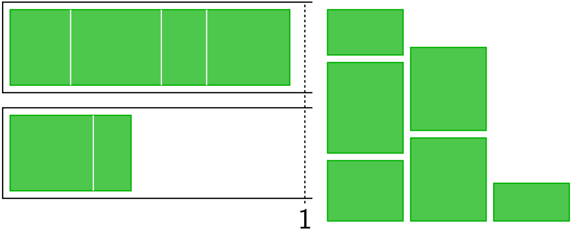

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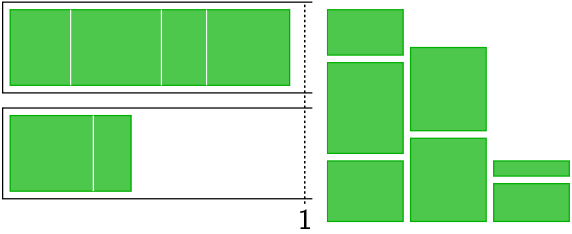

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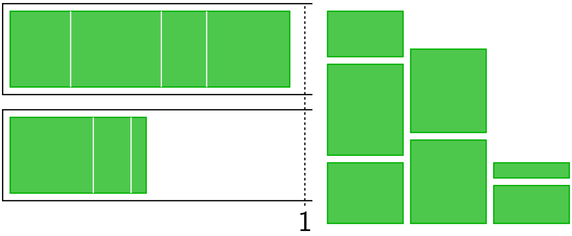

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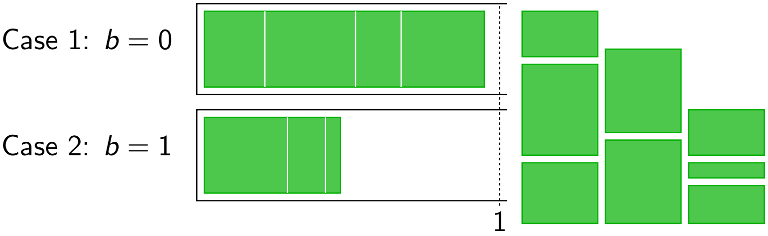

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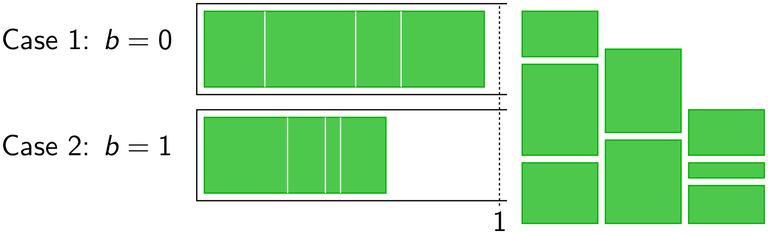

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This is the best you can do for the SKP in randomized online computation.

Conclusion

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General Knapsack Problem

- Not competitive with sub-logarithmic advice
- $(1+\varepsilon)$-competitive with logarithmic advice, $\varepsilon > 0$
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Thank you for your attention!