Near-optimal Algorithms for Online Linear Programming

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Joint work with Shipra Agrawal and Yinyu Ye

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- Applications: revenue management, channel allocation in communication networks, charging allocation for electric vehicles, etc.

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Our objective: to find a decision rule for the online problem such that it achieves *near-optimal* performance, i.e., an algorithm that can achieve values close to OPT.

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The algorithm is evaluated on the expected performance over all the permutations comparing to the offline optimal solution, i.e., an algorithm $\mathcal{A}$ is c-competitive if and only if

$$E_\sigma \left[\sum_{t=1}^{n} \pi_t x_t(\sigma, \mathcal{A}) \right] \geq c \cdot OPT$$

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- Our assumption is strictly weaker than the i.i.d. model

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We propose an algorithm such that for any fixed $\epsilon > 0$*, our online algorithm is* $1 - O(\epsilon)$ *competitive for online linear programming on all inputs when*

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We propose an algorithm such that for any fixed $\epsilon > 0$, our online algorithm is $1 - O(\epsilon)$ competitive for online linear programming on all inputs when

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Theorem

For any online algorithm for the online linear program in random permutation model, there exists an instance such that the competitive ratio is less than $1 - O(\epsilon)$ if

$$B = \min_i B_i \leq \frac{\log(m)}{\epsilon^2}$$

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Our algorithm is quite different from theirs due to the higher dimension

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- We consider a much more general problem
- Their results is weaker by a factor or ϵ and also depend on OPT which is not known until the problem is solved. (Our condition can be verified before solving the problem)

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Dual Optimal Price

- For the offline linear program, there exists a dual price vector $\mathbf{p}^*$ for each goods such that, $x_t^* = 1$ if $\pi_t > \mathbf{a}_t^T \mathbf{p}^*$ and $x_t^* = 0$ otherwise, is near optimal

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We first show a one-time learning algorithm to illustrate this idea. Then we show that we can improve the algorithm by updating the price vector more frequently.

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   Get the optimal dual solution $\hat{\mathbf{p}}$;
3. Determine the future allocation $x_t(\hat{\mathbf{p}})$ as:

$$x_t(\hat{\mathbf{p}}) = \begin{cases} 0 & \text{if } \pi_t \le \hat{\mathbf{p}}^T \mathbf{a}_t \\ 1 & \text{if } \pi_t > \hat{\mathbf{p}}^T \mathbf{a}_t \end{cases}$$

   If $a_{it} x_t(\hat{\mathbf{p}}) \le B_i - \sum_{j=1}^{t-1} a_{ij} x_j$, set $x_t = x_t(\hat{\mathbf{p}})$; otherwise, set $x_t = 0$.

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By the complementarity conditions of LP, if we can show that $\sum_{t=1}^{n} a_{it} x_t(\hat{\mathbf{p}}) = B_i$ for each i, then $x_t(\hat{\mathbf{p}})$ is the optimal solution to the offline problem.

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Lemma

With probability $1 - \epsilon$,

$$(1 - 3\epsilon) B_i \leq \sum_{t=1}^{n} a_{it} x_t(\hat{\mathbf{p}}) \leq B_i, \forall i = 1, \ldots, m$$

given $B \geq \frac{6m \log(n/\epsilon)}{\epsilon^3}$.

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The proof uses intensively concentration inequalities (Hoeffding-Bernstein Inequalities) and union bound arguments

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Then by the above lemmas, it is easy to show that with high probability $\sum_{t=1}^{n} \pi_t x_t(\hat{\mathbf{p}}) \ge (1 - 3\epsilon) OPT$.

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Lemma

Let $OPT(S)$ denote the optimal value of the linear program

$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t \in S} \pi_t x_t & \\ \text{subject to} & \sum_{t \in S} a_{it} x_t \leq \epsilon B_i, & i = 1, ..., m \\ & 0 \leq x_t \leq 1, & t \in S. \end{array}$$

over random sample $S \subset N$ where $|S| = \epsilon |N|$, and $OPT(N)$ denote the optimal value of the original offline linear program. Then,

$$E[OPT(S)] \leq \epsilon OPT(N)$$

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- With high probability, the first ϵ portion of the objective value, a learning cost, doesn't contribute too much.

Therefore, we proved that our one-time learning algorithm is $1 - O(\epsilon)$-competitive if $B \geq \frac{6m \log(n/\epsilon)}{\epsilon^3}$.

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Question

- How often should we update the price?

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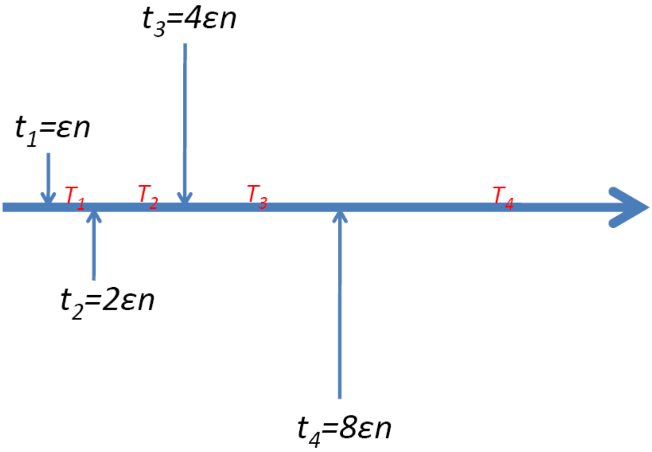
And this price is used to determine the allocation for the next immediate period.

Geometric Pace of Price Updating



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- The numbers h_ℓ play an important role in improving the condition on B in our main theorem. It balances the probability that the inventory constraint ever gets violated and the lost of objective value due to the factor $1 - h_\ell$.

Dynamic Price Updating Algorithm

- In this algorithm, we update the price $\log_2(1/\epsilon)$ times during the entire time horizon.
- The numbers h_ℓ play an important role in improving the condition on B in our main theorem. It balances the probability that the inventory constraint ever gets violated and the lost of objective value due to the factor $1 - h_\ell$.
- Choosing large h_ℓ (more conservative) at the beginning periods and smaller h_ℓ (more risk neutral) at the later periods, we can control the loss of objective value by an ϵ order while the required size of B can be weakened by an ϵ factor.

Proof Outline of the Dynamic Price Updating Algorithm

The proof is similar to the proof of the one-time learning algorithm. We first show the following lemma:

Lemma

For any $\epsilon > 0$, with probability $1 - \epsilon$:

$$\sum_{t=\ell+1}^{2\ell} a_{it} x_t(\hat{\mathbf{p}}^\ell) \leq \frac{\ell}{n} b_i, \quad \textit{for all } i \in \{1, \ldots, m\}, \ \ell \in \{\epsilon n, 2\epsilon n, \ldots\}$$

given $B = \min_i b_i \geq \frac{10 m \log(n/\epsilon)}{\epsilon^2}$.

This lemma states that for each time period, with high probability, the inventory consumed by $x(\hat{\mathbf{p}})$ is less than the proportional total inventory.

Proof Outline of the Dynamic Price Updating Algorithm

Next we need to show that by introducing a factor h_ℓ, the loss of objective value is small.

It is easy to see that at period ℓ, the lost of objective value is about $h_\ell \cdot \frac{\ell}{n} OPT$. Then the total loss of objective value is about

$$\begin{array}{rl} & \sum_{\ell \in \{\epsilon n, 2\epsilon n, \ldots\}} h_\ell \frac{\ell}{n} OPT \\ = & \epsilon \sum_{\ell \in \{\epsilon n, 2\epsilon n, \ldots\}} \sqrt{\frac{\ell}{n}} OPT \\ \leq & \epsilon OPT \sum \sqrt{\frac{1}{2^i}} \\ \leq & O(\epsilon OPT) \end{array}$$

Therefore, the loss of objective value is very small. And we can conclude that this algorithm gives a near-optimal solution.

Summary

We propose a dynamic near-optimal algorithm for a class of online linear programming problems under the random permutation model

- It solves linear programs for dual price based on revealed data, and uses these prices to make future allocations
- The algorithm has the feature of "learning-while-doing", and the pace the price is updated is neither too fast nor too slow
- The application includes various online resource allocation and revenue management problems