DM582 Solutions

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This document contains written solution to exercise problems from the course DM582 (spring 2024). The solutions given here might differ from the solutions discussed in class. In class, we place more emphasis on the intuition leading to the correct answer. Please do not consider reading these solutions an alternative to attending the exercise classes.

References to CLRS refer to the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein.

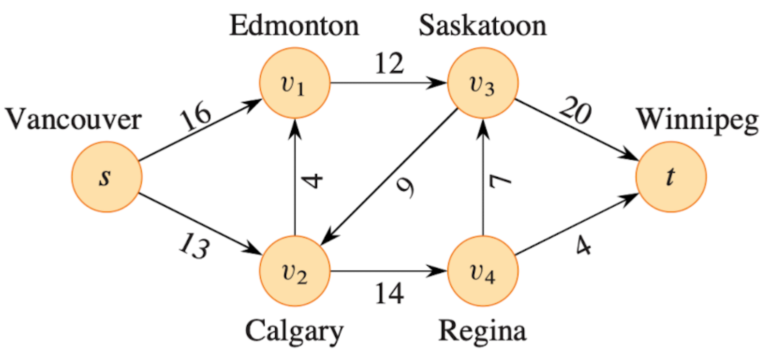
This document will inevitably contain mistakes. If you find some, please report them to me (Mads) so that I can correct them.

Sheet 2

CLRS, 24.2-3

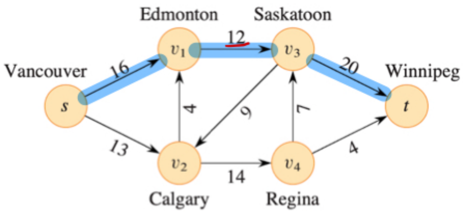
Exercise

Show the execution of the Edmonds-Karp algorithm on the flow network of Figure 24.1(a).



Suggested solution

See next page. I hope you can make out what it says.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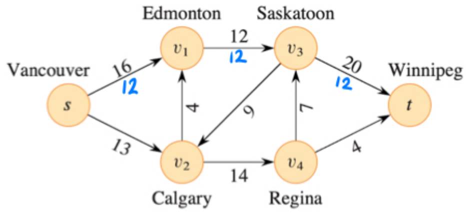
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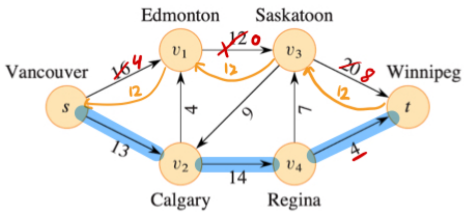
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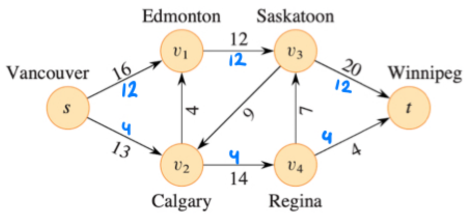
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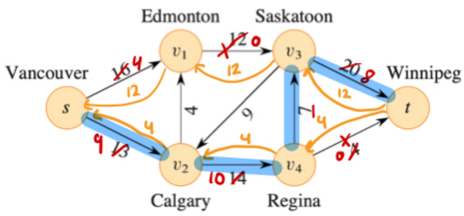
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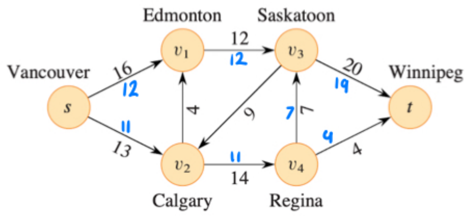
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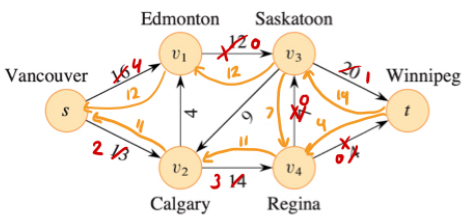
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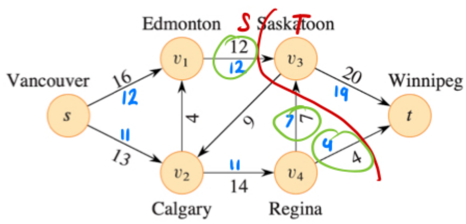
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CLRS, 24.2-7

Exercise

Prove lemma 24.2.

Lemma 24.2. Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function $f_p : V \times V \to \mathbb{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ 0 & \text{otherwise} \end{cases}$$

Then, f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Suggested solution

Note: Recall that an augmenting path is simple by definition.

First, we argue that f_p satisfies flow conservation. To see this, consider an arbitrary vertex $v \in V(G) \setminus \{s, t\}$. If v is not on p then the flow in and out of v is 0. If v is on p then, by the definition of f_p and since p is a simple path, there is exactly one arc uv entering v and exactly one arc vu' leaving v with $f_p(u, v) = f_p(v, u') = c_f(p)$. Hence, flow is conserved at v.

Next, we argue that f_p respects the capacity constraint c_f. An arc uv not on p has no flow and thus satisfies $f_p(u, v) \leq c_f(u, v)$ trivially. Let uv be an arbitrary arc on p. Then

$$\begin{aligned} f_p(u, v) &= c_f(p) \\ &= \min\{c_f(x, y) |\ xy \text{ on } p\} \\ &\leq c_f(u, v) \end{aligned}$$

where the last inequality holds since $c_f(u, v)$ is the residual capacity of a particular arc on p and $c_f(p)$ is the minimum residual capacity taken over all such arcs.

Lastly, we have $|f_p| = c_f(p) > 0$ since no arc in G_f has capacity 0 by definition and exactly one arc sv with non-zero flow leaves s and $f_p(s, v) = c_f(p)$.

CLRS, 24.2-8

Exercise

Suppose that we redefine the residual network to disallow edges into s. Argue that the procedure Ford-Fulkerson still correctly computes a maximum flow.

Suggested solution

The Ford-Fulkerson method computes the maximum flow by finding a sequence of augmenting (s, t)-paths in the residual network. No simple (s, t)-path can use an arc into s (since then it would not be a simple), so no implementation of the Ford-Fulkerson method is affected by the removal of such arcs.

CLRS, 24.2-9

Exercise

Suppose that both f and f' are flows in a flow network. Does the augmented flow $f \uparrow f'$ satisfy the flow conservation property? Does it satisfy the capacity constraint?

Suggested solution

$f \uparrow f'$ does satisfy flow conservation, but the capacity constraints are not at all respected. The definition of $(f \uparrow f')(u, v)$ (24.4) reduces to $(f \uparrow f') = f(u, v) + f'(u, v)$ for all arcs $uv \in E(G)$ since we assume no antiparallel arcs. Thus, for any $v \in V(G)$

$$\begin{aligned}
\sum_{uv \in E(G)} (f \uparrow f')(u, v) &= \sum_{uv \in E(G)} f(u, v) + f'(u, v) \\
&= \sum_{uv \in E(G)} f(u, v) + \sum_{uv \in E(G)} f'(u, v) \\
&= \sum_{vu \in E(G)} f(v, u) + \sum_{vu \in E(G)} f'(v, u) \\
&= \sum_{vu \in E(G)} f(v, u) + f'(v, u) \\
&= \sum_{vu \in E(G)} (f \uparrow f')(v, u).
\end{aligned}$$

We can construct an example with only one arc st showing that the capacity constraints are not respected (we did this in class).

CLRS, 24.2-10

Exercise

Show how to find a maximum flow in a flow network by a sequence of at most $|E|$ augmenting paths. (Hint: Determine the paths after finding the maximum flow.)

Suggested solution

Let f be a maximum flow in G. Take any simple path P from s to t in G using only arcs with positive flow. If no such path exists, then $|f| = 0$ and thus the empty sequence of augmenting paths results in a maximum flow.

Otherwise, let α be the minimum flow on any arc on P and let f' be the flow obtained by subtracting α from the flow on each arc of P. Then f' is a flow in G with $|f'| = |f| - \alpha$. Furthermore, at least one arc (the one with flow α) has no flow in f' and thus will never be used again.

Repeating this process at most $|E|$ we obtain the desired paths.

CLRS, 24.2-11

Exercise

The edge connectivity of an undirected graph is the minimum number k of edges that must be removed to disconnect the graph. For example, the edge connectivity of a tree is 1, and the edge connectivity of a cyclic chain of vertices is 2. Show how to determine the edge connectivity of an undirected graph $G = (V, E)$ by running a maximum-flow algorithm on at most $|V|$ flow networks, each having $O(V + E)$ vertices and $O(E)$ edges.

Suggested solution

Call a set of edges whose removal disconnects G a *cut-set*. Let k be the size of a minimum cut-set in G.

Obtain G' by replacing every edge in G by a pair of antiparallel arcs and split one of the arcs into two as described on page 673. Set the capacity of all arcs to 1 and pick an arbitrary vertex $s \in V(G)$ as the source. For every $t \in V(G) \setminus \{s\}$, compute the value of a maximum flow f_{st} in G' from s to t. Let $f_{\min}$ be the smallest value of $|f_{st}|$ obtained for all $t \in V(G)$. We show that $f_{\min} \leq k$ and $k \leq f_{\min}$, from which we conclude $f_{\min} = k$.

The minimum cut-set separates G into two at least two[1] components S and T. Without loss of generality assume that $s \in V(S)$ and pick an arbitrary $t \in V(T)$. There are exactly k arcs from S to T in G', so $f_{\min} \leq |f_{st}| \leq k$ since all arcs have capacity 1.

Conversely, by the max-flow min-cut theorem, for any max-flow f_{st} there is an (s, t)-cut (S, T) with capacity $|f_{st}|$. Since the capacity of all arcs is 1, there are exactly $|f_{st}|$ arcs from S to T in G'. The removal of the k corresponding edges in G disconnects G and thus $k \leq |f_{st}|$ for any $t \in V(G)$. In particular, $k \leq f_{\min}$.

Since $f_{\min} \leq k$ and $k \leq f_{\min}$, we conclude $f_{\min} = k$.

[1] In fact, exactly 2. Assume more and obtain a contradiction to the minimality of the cut-set.

CLRS, 24.2-12

Exercise

You are given a flow network G, where G contains edges entering the source s. Let f be a flow in G with $|f| \geq 0$ in which one of the edges (v, s) entering the source has $f(v, s) = 1$. Prove that there must exist another flow f' with $f'(v, s) = 0$ such that $|f| = |f'|$. Give an $O(E)$-time algorithm to compute f', given f and assuming that all edge capacities are integers.

Suggested solution

We assume that f is an integer flow.

Find a simple 'path' p from s back to s in the residual network starting with the edge sv and augment the flow along this path by $1 \leq c_f(p)$ (which holds since f is an integer flow). s loses 1 unit of flow out and one unit of flow in, so the flow value remains the same.

CLRS, 24.2-13

Exercise

Suppose that you wish to find, among all minimum cuts in a flow network G with integer capacities, one that contains the smallest number of edges. Show how to modify the capacities of G to create a new flow network G' in which any minimum cut in G' is a minimum cut with the smallest number of edges in G.

Suggested solution

We obtain G' by adding a sufficiently small quantity $\epsilon > 0$ to the capacity of each arc (we will see how small). Let C be the capacity of a minimum cut and let m be the number of edges in a minimum cut with the fewest number of edges. If a minimum cut (S, T) contains $M > m$ edges, then the capacity of (S, T) in G' is $C + M\epsilon > C + m\epsilon$ and thus (S, T) is not a minimum cut in G'. Note that this holds for any $\epsilon > 0$.

If we pick ϵ sufficiently small, no cut that was previously not a minimum cut will become a minimum cut. Indeed, since all capacities are integers, a minimum cut in G has capacity at least 1 less than any non-minimum cut. Thus, if we pick ϵ such that the capacity of a minimum cut in G increases by less than 1, we obtain the desired property.

We show that $\epsilon < \frac{1}{|E|}$ is sufficiently small (in practice, we could pick $\epsilon = \frac{1}{|E|+1}$). Let (S, T) be a minimum cut in G with capacity C and m edges. Let C' be the capacity of (S, T) in G'. Then

$$C' = C + m\epsilon \leq |E|\epsilon < C + 1$$

since no cut contains more than $|E|$ edges.