Meldable Priority Queues

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About These Slides

Disclaimer

These slides contain much more text than I usually put on slides.

The reason is that no good text exists for this material at this level. So, the slides should replace a textbook.

Thus, the slides will be less suited for lecturing.

Priority Queues

A priority queue is a data type for a collection of elements, each of which has an associated priority.

The minimal set of operations provided for a priority queue are the following:

`q = PriorityQueue()`: Initializes an empty priority queue.

`q.insert(e, p)`: Inserts the element `e` with priority `p` into `q`.

`q.findMin()`: Returns the element of highest priority (traditionally indicated by smallest value) in `q`.

`q.deleteMin()`: Deletes and returns the element of highest priority from `q`.

A priority queue may have additional operations such as `decreaseKey`, `meld`, and others.

Priority Queues

The most well-known implementation of the priority queue data type is the *binary heap* data structure.

A binary heap provides `findMin` in $O(1)$ time and `insert` and `deleteMin` in $O(\log n)$, where n is the number of elements in the priority queue when the operation is carried out.

We will be interested in the operation `meld`.

`meld(q, p)`: Returns a new priority queue containing all the elements from `q` and `p` (destructive).

The standard binary heap implementation cannot provide an efficient implementation of this operation.

Leftist Heaps [Crane, Stanford, 1972]

A leftist heap is implemented as an *annotated binary tree*.

Each node contains an element with a priority (we just show the priority of the element) and a rank.

The tree is *heap-ordered*, i.e., the priority of a node is at most the priority of its children.

The *rank* is defined as the distance to *nil*[1] in the following sense:

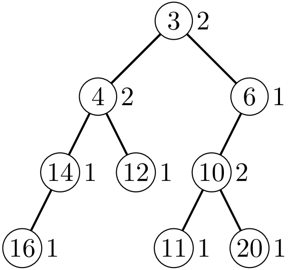
Think of a nil reference as a reference to a special node with rank zero. Then a node containing a nil reference has rank one. Other nodes have rank one plus the minimum of the ranks of its children.

The tree must be *leftist*, which we define to mean that for any node `u`,

$$\texttt{u.left().rank()} \geq \texttt{u.right().rank()}$$

[1] Or *None*, *null*, or some other name for an initialized missing reference.

Example Leftist Heap



Melding Two Leftist Heaps

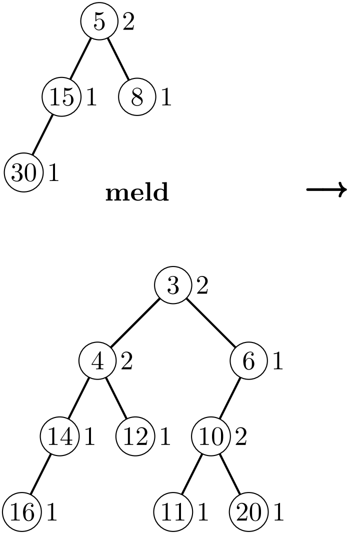
We carry out a meld as follows:

1. Merge the right-most paths of the two argument heaps according to the priorities via their right child references.
2. Adjust the ranks bottom-up on the right-most path in the result.
3. Switch the children of nodes on the right-most path if the leftist requirement is violated.

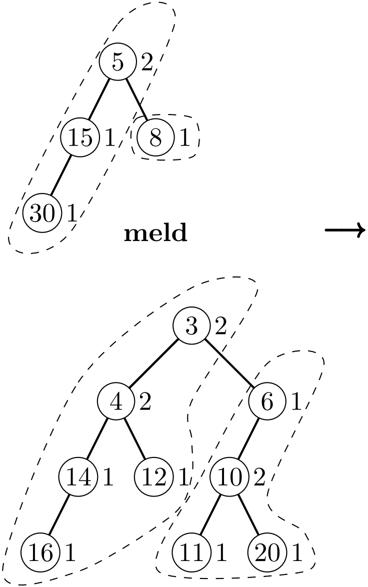
The result is clearly a leftist heap.

It takes time proportional to the sum of the lengths of the arguments right-most paths.

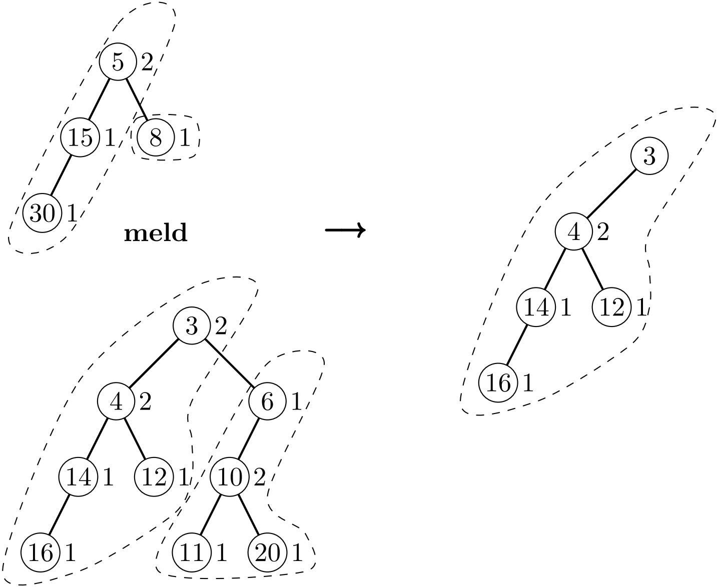
Leftist Heaps



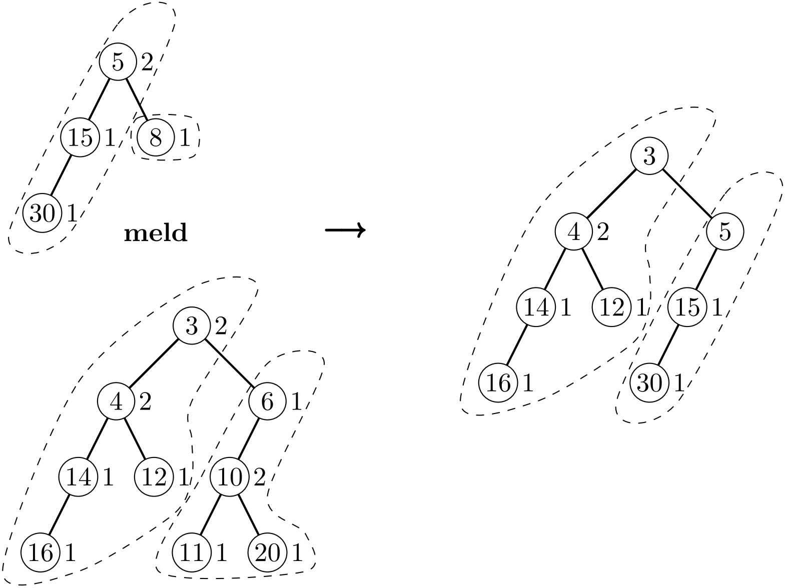
Leftist Heaps



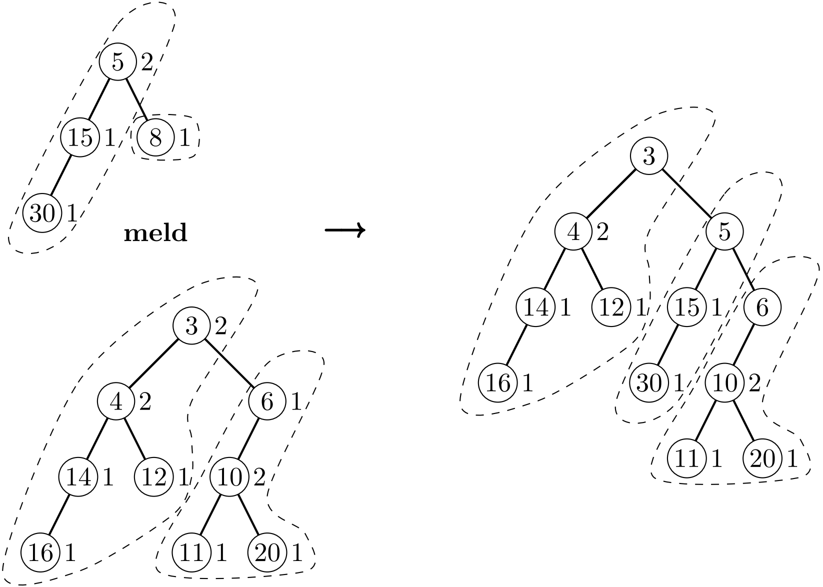
Leftist Heaps



Leftist Heaps



Leftist Heaps



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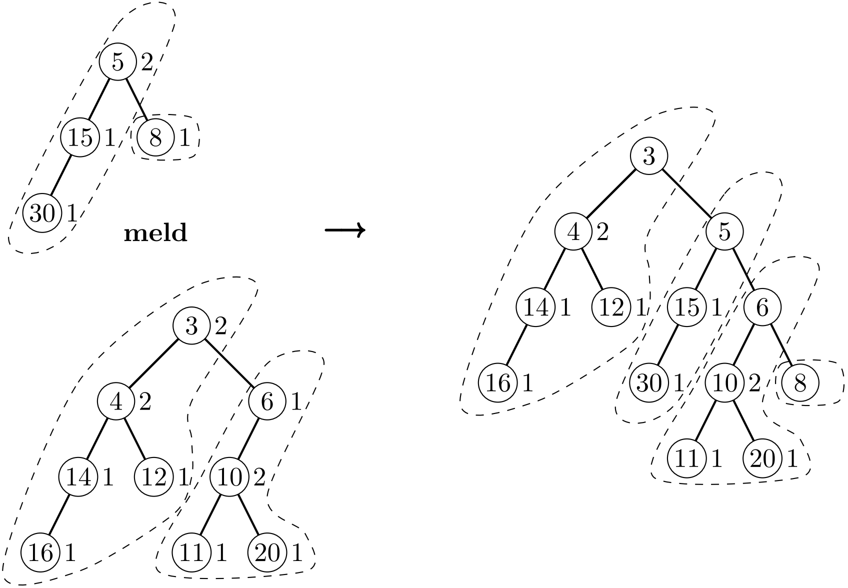

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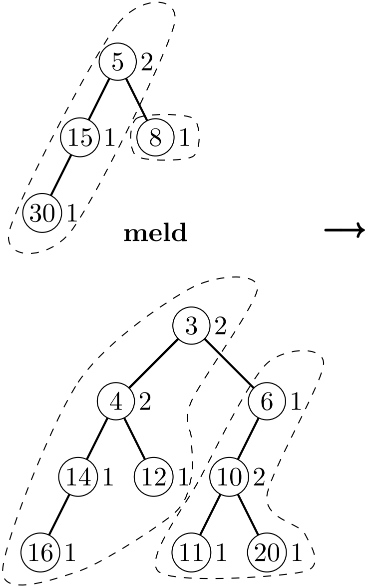

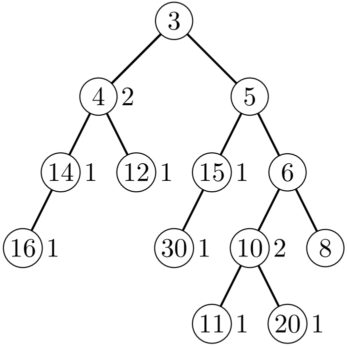

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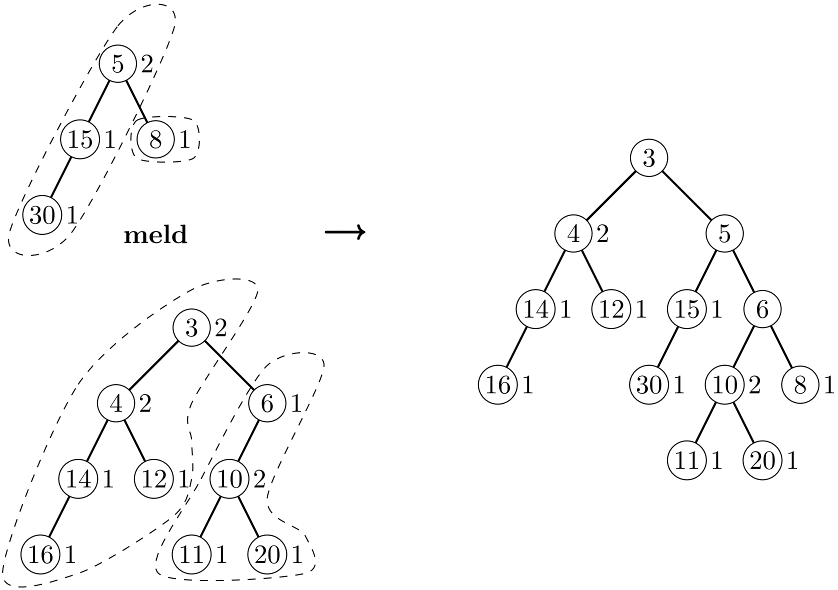

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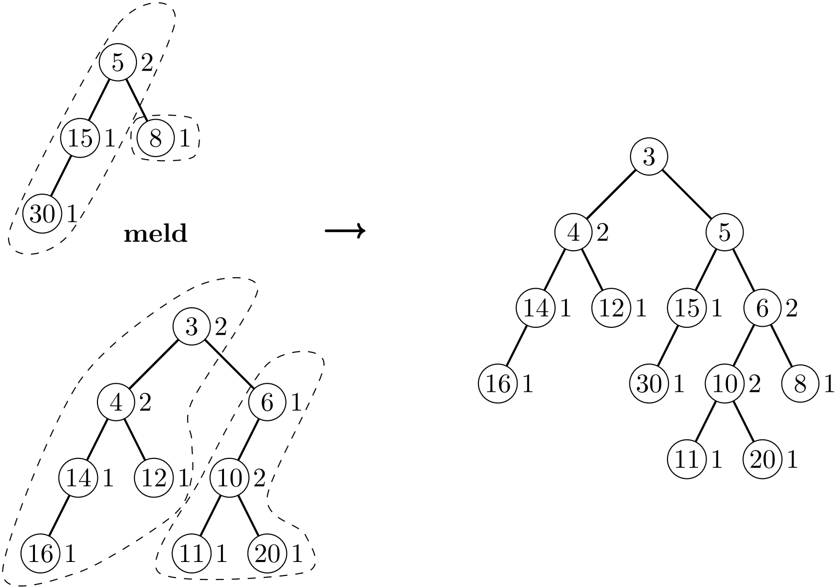

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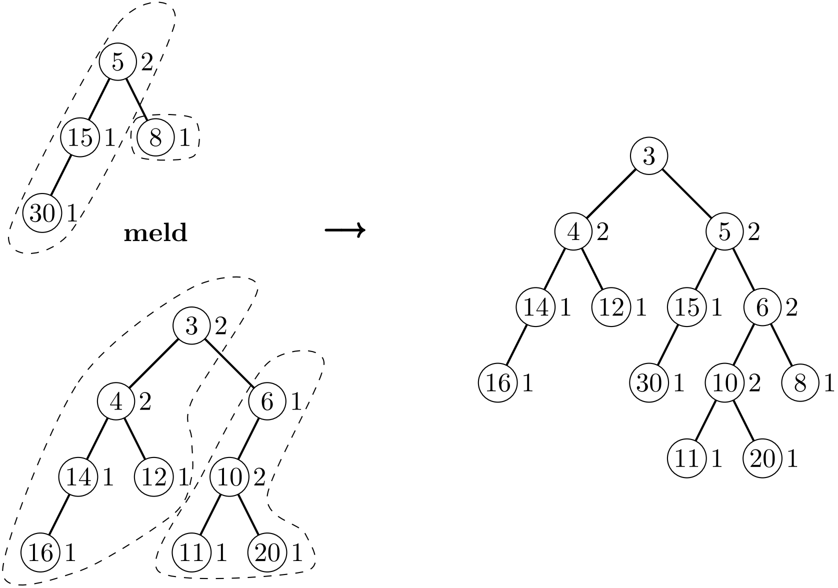

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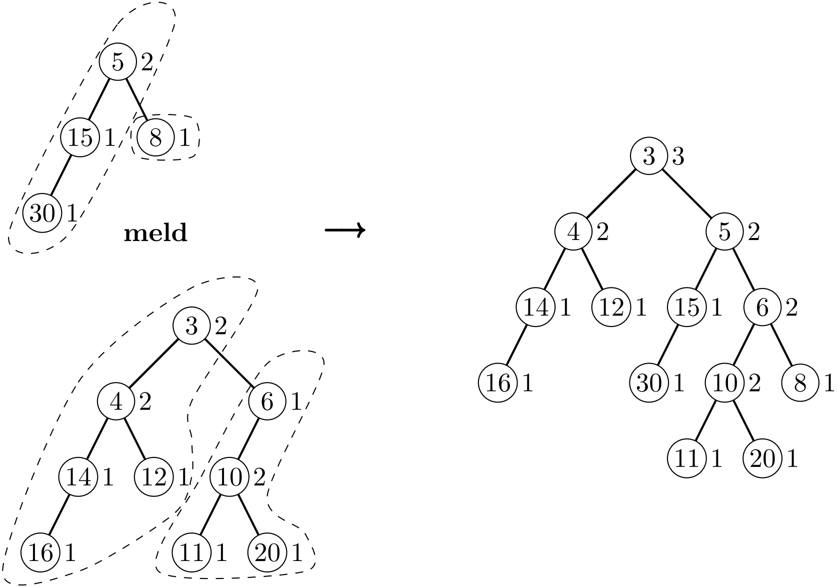

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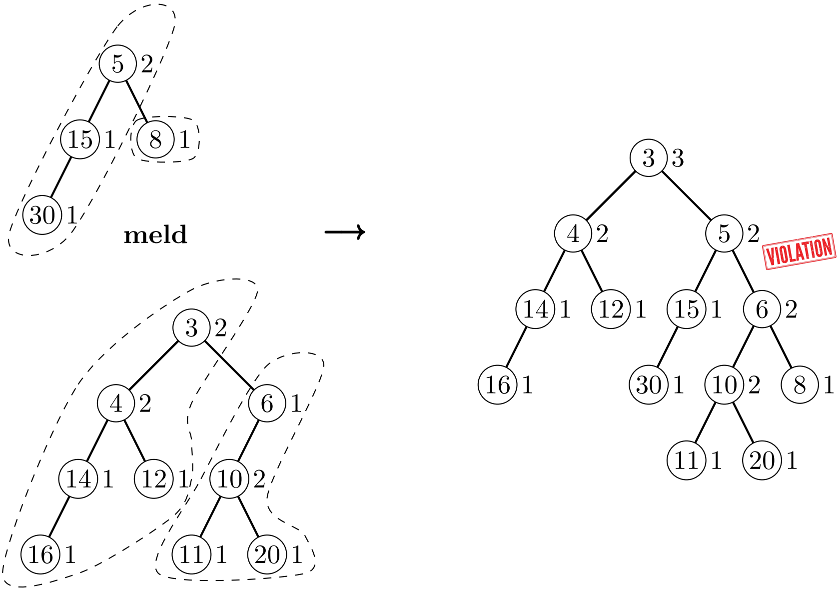
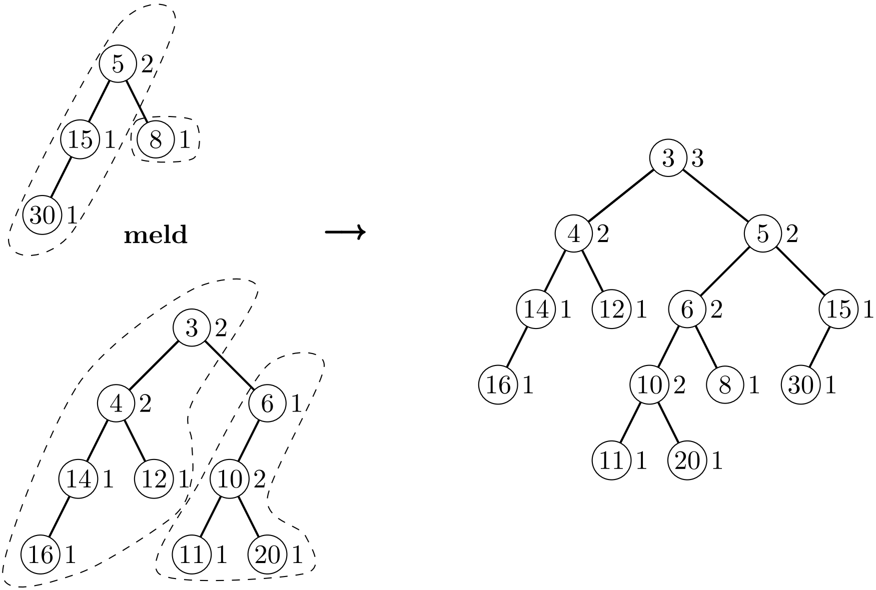


Leftist Heaps



Leftist Heap Complexity

SDU

Lemma

In a leftist tree, the subtree of a node with rank r contains at least $2^r - 1$ nodes.

Proof By structural induction. For the base case, a node with no children has rank 1 and its subtree contains $2^1 - 1 = 1$ nodes. For the induction step, a node cannot have rank r unless both of its children have rank at least $r - 1$. By induction, its subtree has at least $2(2^{r-1} - 1) + 1 = 2^r - 1$ nodes. □

Corollary

The maximal rank of the root of a leftist heap with n elements is $\log(n + 1)$.

Proof Let r be the rank of the root. By the above lemma, $n \geq 2^r - 1$, so $r \leq \log(n + 1)$. □

Leftist Heap Complexity

SDU

Theorem

A `meld` of two leftist heaps with n_1 and n_2 elements takes time $O(\log n)$, where $n = n_1 + n_2$.

Proof For any node of rank r with left and right children ranks of r_l and r_r, since $r_l \geq r_r$ (the leftist property), $r = r_r + 1$. Thus, there are exactly r nodes on the right-most path of a root with rank r.

The time to meld the two heaps is proportional to the sum of the lengths of the two right-most paths, which amounts to at most

$$\log(n_1 + 1) + \log(n_2 + 1) \leq 2\log(\max\{n_1, n_2\} + 1) \leq 2\log n.$$

$\square$

Leftist Heap Operations

Operations other than `meld` are either trivial or can be reduced to `meld`, so we get the following results are corollaries.

`q = PriorityQueue()`: Clearly $O(1)$.

`q.insert(e, p)`: Make singleton heap and meld with `q` in $O(\log n)$.

`q.findMin()`: Clearly $O(1)$.

`q.deleteMin()`: Remove the root, meld its two children in $O(\log n)$.

`q = buildHeap(elements)`: Notice that the shape of a classic heap makes it a leftist heap that we can annotate with ranks in linear time and get this operation in $O(n)$.

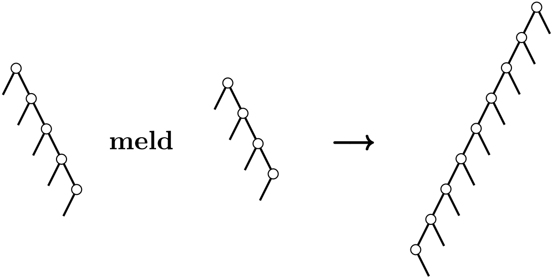
Skew Heaps [Sleator & Tarjan]

We try to do as well or better with less information!

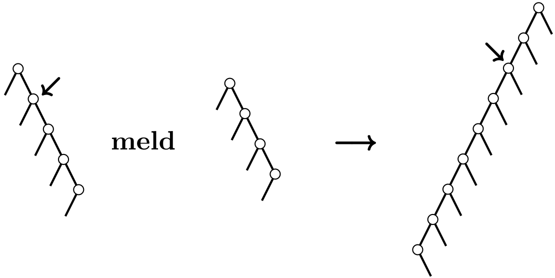
A skew heap is mostly the same as a leftist heap, but we do not keep any rank information. Instead, after merging the right-most paths according to priorities, we switch the subtrees of every node on that path!

So, the two right-most paths become one left-most path.

Skew Heaps Example



Skew Heaps Example

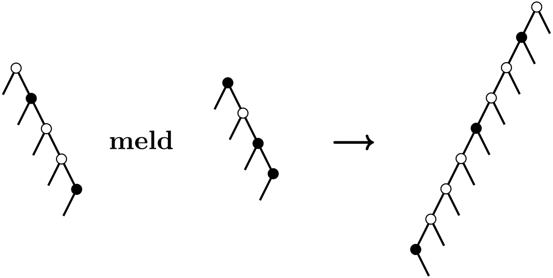


Skew Heaps Analysis

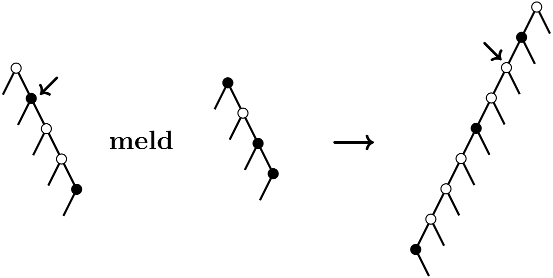
A node is *heavy* ($\bullet$) if its right subtree contains more nodes than its left subtree. Otherwise, it is called *light* ($\circ$).

During the merge and the switches, nodes on the right-most paths before the `meld` can change status from heavy to light or light to heavy.

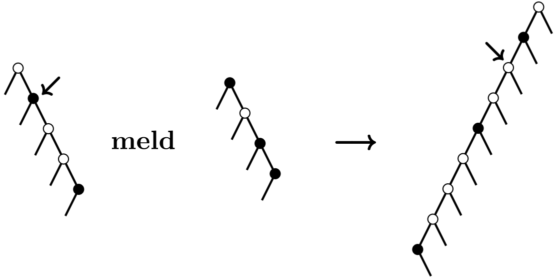
Skew Heaps



Skew Heaps



Skew Heaps

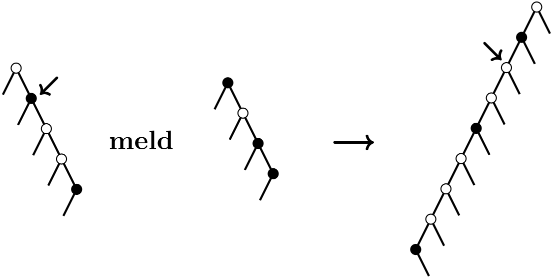


During a merge, a heavy node may be come even heavier!

So, when we switch the subtrees, it will definitely become light.

We do not know if a light node changes status or not.

Skew Heaps



During a merge, a heavy node may be come even heavier!

So, when we switch the subtrees, it will definitely become light.

We do not know if a light node changes status or not.

heavy	$\rightarrow$	light
light	$\rightarrow$	?

Skew Heaps

Lemma

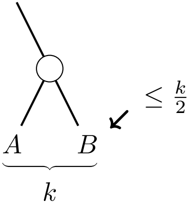
There are at most $\log n$ light nodes on the right-most path of a skew heap.

Proof A heavy node would have $|B| > |A|$.

But it is light, so $|B| \leq |A|$.

Thus, traversing the right-most path from root to leaf, considering the number of nodes in A and B, we always move towards the subtree with at most half of the nodes.

This can only happen $\log n$ times.



$\square$

Skew Heaps

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Theorem

For skew heaps, `meld` is $O_A(\log n)$ (amortized $O(\log n)$).

Proof Let l_i and h_i denote the number of light and heavy nodes, respectively, on the right-most path of argument i, $i \in \{1, 2\}$.

As for leftist heaps, the cost of `meld` is $(l_1 + h_1) + (l_2 + h_2)$.

Define the potential function $\Phi(T)$ to be the number of heavy nodes in T. This is initially zero and always non-negative, so results are valid.

In the worst case, all the light nodes become heavy so we need to pay into the potential for them.

Operation	Cost	$\Delta\Phi$	Amortized Cost
`meld`	$(l_1 + h_1) + (l_2 + h_2)$	$-h_1 - h_2 + l_1 + l_2$	$2(l_1 + l_2)$

The result follow by the lemma. $\square$

Skew Heaps

As for leftist heaps, all the other operations follow.

`q = PriorityQueue()`: Clearly $O(1)$.

`q.insert(e, p)`: Make singleton heap and meld with `q` in $O_A(\log n)$.

`q.findMin()`: Clearly $O(1)$.

`q.deleteMin()`: Remove the root, meld its two children in $O_A(\log n)$.

`q = buildHeap(elements)`: Notice that the shape of a classic heap makes all nodes light, so we can perform the operation in $O(n)$ and the potential is zero, so the amortized results for the above operations hold.

References I

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