DM582 Exercises - Sheet 1

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February 10, 2025

This document contains exercises from the course DM582 (spring 2025). Most exercises are from the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein (CLRS), the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos (KT), and the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

References to CLRS refer to the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein.

References to KT refer to the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos.

References to Rosen refer to the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

This document will inevitably contain mistakes. If you find some, please report them to your TA so that we can correct them.

Sheet 1

CLRS, 24.1-3

Exercise

Suppose that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \leadsto v \leadsto t$ for all vertices $v \in V$. Let v be a vertex for which there is no path $s \leadsto v \leadsto t$. Show that there must exist a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $u \in V$.

Note: CLRS defines a path as a sequence of not necessarily distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for all $i = 1, 2, \ldots, k-1$. Importantly for this exercise, this means that a path from s to v to t may visit some vertex multiple times. In other books, you might find that such a sequence is called a walk.

CLRS, 24.1-4

Exercise

Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted αf, is a function from $V \times V$ to $\mathbb{R}$ defined by

$$(\alpha f)(u, v) = \alpha f(u, v).$$

Prove that the flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all α in the range $0 \leq \alpha \leq 1$.

CLRS, 24.1-6

Exercise

Professor Adam has two children who, unfortunately, dislike each other. The problem is so severe that not only do they refuse to walk to school together, but in fact each one refuses to walk on any block that the other child has stepped on that day. The children have no problem with their paths crossing at a corner. Fortunately both the professor's house and the school are on corners, but beyond that he is not sure if it is going to be possible to send both of his children to the same school. The professor has a map of his town. Show how to formulate the problem of determining whether both his children can go to the same school as a maximum-flow problem.

CLRS, 24.1-7

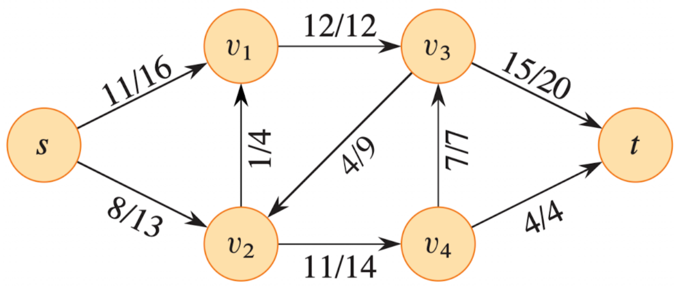
Exercise

Suppose that, in addition to edge capacities, a flow network has vertex capacities. That is, each vertex has a limit on how much flow can pass through. Show how to transform a flow network with vertex capacities into an equivalent flow network without vertex capacities, such that a maximum flow in the original network has the same value as a maximum flow in the transformed network. How many vertices and edges does the transformed network have?

CLRS, 24.2-2

Exercise

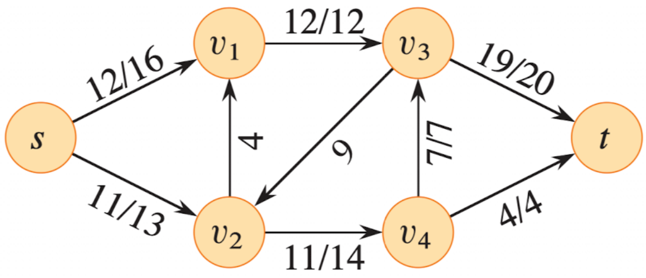
In Figure 24.1(b), what is the net flow across the cut $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$? What is the capacity of this cut? The below is the figure being referenced.



CLRS, 24.2-4

Exercise

In the example of Figure 24.6, what is the minimum cut corresponding to the maximum flow shown? Of the augmenting paths appearing in the example, which one cancels flow? The below is the figure being referenced.



CLRS, 24.2-5

Exercise

The construction in Section 24.1 to convert a flow network with multiple sources and sinks into a single-source, single-sink network adds edges with infinite capacity. Prove that any flow in the resulting network has a finite value if the edges of the original network with multiple sources and sinks have finite capacity.

CLRS, Problem 24.4

Exercise

Let $G = (V, E)$ be a flow network with source s, sink t, and integer capacities. Suppose that you are given a maximum flow in G.

a. Suppose that the capacity of a single edge $(u, v) \in E$ increases by 1. Give an $O(V + E)$-time algorithm to update the maximum flow.

b. Suppose that the capacity of a single edge $(u, v) \in E$ decreases by 1. Give an $O(V + E)$-time algorithm to update the maximum flow.