DM582 Exercises - Sheet 1

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This document contains exercises from the course DM582 (spring 2025). Most exercises are from the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein (CLRS), the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos (KT), and the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

The solutions given here might differ from the solutions discussed in class. In class, we place more emphasis on the intuition leading to the correct answer. Please do not consider reading these solutions an alternative to attending the exercise classes.

References to CLRS refer to the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein.

References to KT refer to the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos.

References to Rosen refer to the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

This document will inevitably contain mistakes. If you find some, please report them to your TA so that we can correct them.

Sheet 1

CLRS, 24.1-3

Exercise

Suppose that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \leadsto v \leadsto t$ for all vertices $v \in V$. Let v be a vertex for which there is no path $s \leadsto v \leadsto t$. Show that there must exist a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $u \in V$.

Note: CLRS defines a path as a sequence of not necessarily distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for all $i = 1, 2, \ldots, k - 1$. Importantly for this exercise, this means that a path from s to v to t may visit some vertex multiple times. In other books, you might find that such a sequence is called a walk.

Suggested solution

Let G be a flow network and let f be a maximum flow in G. Suppose that there is no path $s \leadsto v \leadsto t$ for some vertex v.

Then either there is no (s, v)-path or there is no (v, t)-path. Note that this is not necessarily true if a path cannot use the same vertex twice.

Suppose there is no (s, v)-path and let U be the set of vertices that can reach v. Then $s \notin U$. If $t \in U$ then s cannot reach t and the zero flow is maximum, so suppose $t \notin U$. There is no arc xy entering U since then also x would be able to reach v and thus be in U by definition. Thus, there is no flow entering U and by flow conservation no arc leaving U has any flow. Hence, setting the flow to 0 on all arcs with at least one endpoint in U does not change the value of f and satisfies $f(u, v) = f(v, u) = 0$ for all $u \in V$.

A very similar argument applies if there is no (v, t)-path. Now, define U to be the set of vertices that are reachable from v. Then $t \notin U$. If $s \in U$ then s cannot reach t and thus the zero flow is maximum, so suppose $s \notin U$. There is no arc leaving U and thus no flow entering U. Hence, setting the flow to 0 on all arcs with at least one endpoint in U we again obtain the desired maximum flow.

CLRS, 24.1-4

Exercise

Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted αf, is a function from $V \times V$ to $\mathbb{R}$ defined by

$$(\alpha f)(u, v) = \alpha f(u, v).$$

Prove that the flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all α in the range $0 \leq \alpha \leq 1$.

Suggested solution

Let $0 \leq \alpha \leq 1$ be a real number and let f_1 and f_2 be feasible flows in a network $G = (V, E)$ with capacity function c. Let $f = \alpha f_1 + (1 - \alpha) f_2$. We show that f is also feasible in G. We must verify that

- $0 \leq f(u, v) \leq c(u, v)$ for all $uv \in E$ and
- $\sum_{vu \in E} f(v, u) = \sum_{uv \in E} f(u, v)$ for all $v \in V \setminus \{s, t\}$. That is, flow is conserved.

Let $uv \in E$ be arbitrary. We first observe that $f(u, v) \geq 0$ since it is the sum of positive numbers. Since f_1 and f_2 are feasible we see that

$$\begin{aligned} f(u, v) &= \alpha f_1(u, v) + (1 - \alpha) f_2(u, v) \\ &\leq \alpha c(u, v) + (1 - \alpha) c(u, v) \\ &= (\alpha + 1 - \alpha) c(u, v) \\ &= c(u, v), \end{aligned}$$

so also $f(u, v) \leq c(u, v)$. We now verify that flow is conserved. Let $v \in V \setminus \{s, t\}$ be arbitrary. We use the fact that f_1 and f_2 are feasible and the definition of f. We obtain

$$\begin{aligned}
\sum_{vu\in E} f(v,u) &= \sum_{vu\in E} \left(\alpha f_1(v,u) + (1-\alpha) f_2(v,u)\right) \\
&= \sum_{vu\in E} \alpha f_1(v,u) + \sum_{vu\in E} (1-\alpha) f_2(v,u) \\
&= \alpha \left(\sum_{vu\in E} f_1(v,u) \right) + (1-\alpha) \left(\sum_{vu\in E} f_2(v,u) \right) \\
&= \alpha \left(\sum_{uv\in E} f_1(u,v) \right) + (1-\alpha) \left(\sum_{uv\in E} f_2(u,v) \right) \\
&= \sum_{uv\in E} \left(\alpha f_1(u,v) + (1-\alpha) f_2(u,v)\right) \\
&= \sum_{uv\in E} f(u,v)
\end{aligned}$$

as desired.

CLRS, 24.1-6

Exercise

Professor Adam has two children who, unfortunately, dislike each other. The problem is so severe that not only do they refuse to walk to school together, but in fact each one refuses to walk on any block that the other child has stepped on that day. The children have no problem with their paths crossing at a corner. Fortunately both the professor's house and the school are on corners, but beyond that he is not sure if it is going to be possible to send both of his children to the same school. The professor has a map of his town. Show how to formulate the problem of determining whether both his children can go to the same school as a maximum-flow problem.

Suggested solution

Let $G = (V, E)$ be a graph representing the map of the town where corners are taken as vertices and streets connecting the corners as arcs. If this gives rise to antiparallel arcs, then these may be handled as described on page 673 of CLRS.

We set the capacity of every arc $uv \in E$ to 1, let s be the vertex representing the house and t be the vertex representing the school. We claim that the children can go to the same school if and only if the value of a maximum flow in G is at least 2.

Suppose that the children can go the same school. That is, there are two arc-disjoint paths from s to t. Sending one unit of flow along each path gives a flow of value 2, so a maximum flow in G has value at least 2.

Conversely, let f be a maximum flow in G and suppose $|f| \geq 2$. [1]

We now construct two arc-disjoint paths P_1 and P_2 from s to t. Since there is flow from s to t, there must be a path P_1 from s to t with positive flow. Since f is an integer flow, every arc on P_1 has flow at least 1. Since, the capacity of every arc is 1, the flow along each arc on P_1 is exactly 1. Thus, we may obtain a new feasible flow f' with $|f'| = |f| - 1$ by setting $f'(u, v) = f(u, v) - 1 = 0$ for all $uv \in A(P_1)$. As $|f| \geq 2$ we have $|f'| \geq 1$, and thus we may repeat this process to find a path P_2 from s to t with positive flow. Note that P_2 cannot use any arc that P_1 uses since $f'(u, v) = f(u, v) - 1 = 0$ for all $uv \in A(P_1)$. Hence, P_1 and P_2 are arc-disjoint as desired.

[1] We need all flow values to be integer. It is okay to assume this, but we ignore it for now. We note that we may obtain such a flow by letting f be constructed via the Ford-Fulkerson method.

CLRS, 24.1-7

Exercise

Suppose that, in addition to edge capacities, a flow network has vertex capacities. That is, each vertex has a limit on how much flow can pass through. Show how to transform a flow network with vertex capacities into an equivalent flow network without vertex capacities, such that a maximum flow in the original network has the same value as a maximum flow in the transformed network. How many vertices and edges does the transformed network have?

Suggested solution

We can use *vertex splitting*. Let $G = (V, E)$ be a network with vertex capacities given by a function $g : V \to \mathbb{R}$. We obtain a network $G' = (V', E')$ without vertex capacities such that a maximum flow in G' has the same value as a maximum flow in G.

For each $v \in V \setminus \{s, t\}$, we split v into two new vertices v_a and v_b such that all arcs entering v now enter v_a and all arcs leaving v now leave v_b. Furthermore, we add an arc $v_a v_b$ from v_a to v_b with capacity $c'(v_a, v_b) = g(v)$. All other capacities remain the same.

Now, let f be any feasible flow in G respecting the vertex capacities. We obtain a feasible flow f' in G' with $|f'| = |f|$ by letting $f'(u_b, v_a) = f(u, v)$ for all arcs $uv \in E$ and $f(v_a, v_b) = \sum_{uv \in E} f(u, v)$.

Since f respects the vertex capacities we have $\sum_{uv \in E} f(u, v) \leq g(v)$ for all $v \in V$ and thus also $f'(v_a, v_b) \leq g(v) = c(v_a, v_b)$. Since f also respects the arc capacities $f(u, v) \leq c(u, v) = c'(u_b, v_a)$ for any $uv \in E$ and thus $f'(u_b, v_a) \leq c'(u_b, v_a)$. Thus, f' is a feasible flow in G'.

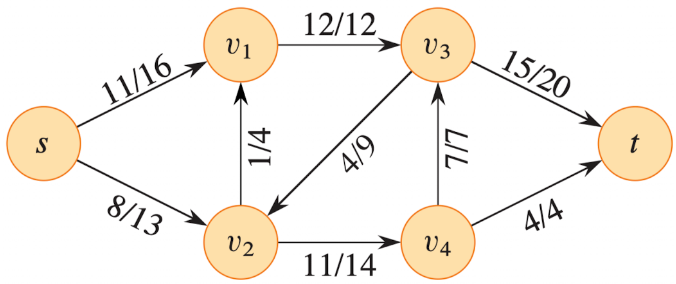
Similarly, one can construct a feasible flow f in G given a feasible flow f' in G' with $|f| = |f'|$ by simply contracting each pair v_a, v_b to a single vertex. This flow will respect the vertex capacities since for any vertex $v \in V$ the flow entering v_a in G' will have to pass through $v_a v_b$ which has capacity $g(v)$.

The new network G' has $2|V| - 2$ vertices and $|E| + |V| - 2$ edges.

CLRS, 24.2-2

Exercise

In Figure 24.1(b), what is the net flow across the cut $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$? What is the capacity of this cut? The below is the figure being referenced.



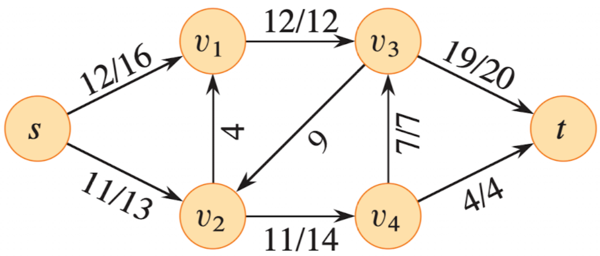
Suggested solution

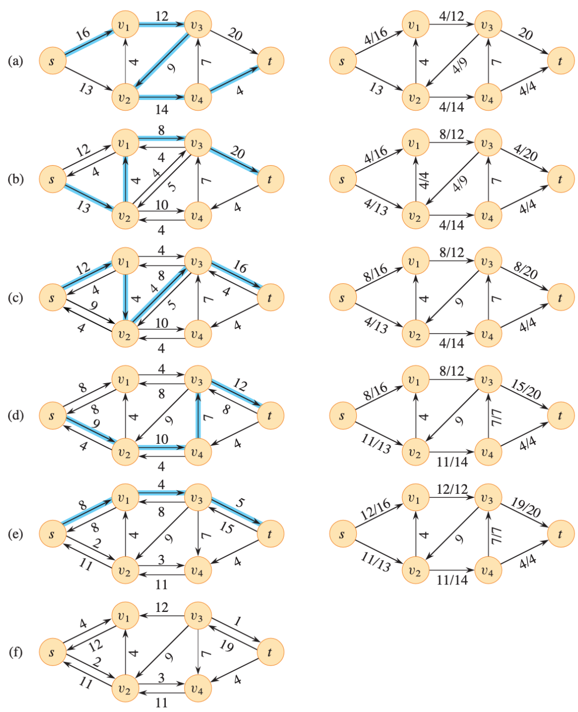
The flow across the cut is $11 + 1 - 4 + 7 + 4 = 19$. The capacity of the cut is $16 + 4 + 7 + 4 = 31$. Recall that the intended meaning of 'the capacity of the (s, t)-cut (S, T)' is the maximum amount of flow that we could possibly send from S to T. Thus, we only consider arcs from S to T when calculating the capacity.

CLRS, 24.2-4

Exercise

In the example of Figure 24.6, what is the minimum cut corresponding to the maximum flow shown? Of the augmenting paths appearing in the example, which one cancels flow? The below is the figure being referenced.





Suggested solution

The cut $(\{s, v_1, v_2, v_4\}, \{v_3, t\})$ is a minimum cut with capacity equal to the value of the indicated flow. The augmenting path shown in subfigure (c) uses the arc v_1v_2, which cancels the flow along the arc v_2v_1 in the original network.

CLRS, 24.2-5

Exercise

The construction in Section 24.1 to convert a flow network with multiple sources and sinks into a single-source, single-sink network adds edges with infinite capacity. Prove that any flow in the resulting network has a finite value if the edges of the original network with multiple sources and sinks have finite capacity.

Suggested solution

Let f be a feasible flow in the modified network with supersource s and supersink t. Suppose some arc su leaving the supersource s has non-finite flow. By flow conservation and since the sum of finite numbers is finite, there must be an arc uv leaving u with non-finite flow. Since the only arcs with infinite capacity are those leaving s or entering t and $u \neq s$, we must have $v = t$. But then u is both a source and a sink in the original network, which is a contradiction. Similarly, if some arc vt entering the supersink t has non-finite flow, then there must be an arc uv entering v with non-finite flow. Since $v \neq t$, we must have $u = s$ and thus v is both a source and a sink in the original network, which is again a contradiction.

CLRS, Problem 24.4

Exercise

Let $G = (V, E)$ be a flow network with source s, sink t, and integer capacities. Suppose that you are given a maximum flow in G.

a. Suppose that the capacity of a single edge $(u, v) \in E$ increases by 1. Give an $O(V + E)$-time algorithm to update the maximum flow.

b. Suppose that the capacity of a single edge $(u, v) \in E$ decreases by 1. Give an $O(V + E)$-time algorithm to update the maximum flow.

Suggested solution

For the sake of simplicity we assume that the given maximum flow f is an integer flow and that no arcs entering s have flow.

In both cases, the algorithms given will execute BFS a constant number of times and otherwise use only constant time. This gives the desired runtime of $O(V + E)$.

a. We start by observing that the value of a maximum flow can increase by at most 1 as a result of the increase. Thus, updating the residual network accordingly and doing a single iteration of the Ford-Fulkerson method (e.g. using BFS) will either result in augmenting the flow by at least 1 or concluding that the flow is already maximum (if there is no (s, t)-path in the residual network).

b. If $f(u, v) \leq c(u, v) - 1$, then no change is required to make the flow feasible after decreasing the capacity of uv. Since the value of a minimum cut cannot increase by decreasing the capacity of an arc, we conclude that the same flow is still maximum.

   Otherwise, find an (s, u)-path and a (v, t)-path using arcs with flow at least 1 and decrease the flow along these paths by 1. This will decrease the flow value by 1 and the flow will now be feasible. Finally, run a single iteration of the Ford-Fulkerson method. This results in augmenting the flow by 1 or we conclude that the modified flow of value $|f| - 1$ is already maximum.