DM582 Exercises - Sheet 2

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This document contains exercises from the course DM582 (spring 2025). Most exercises are from the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein (CLRS), the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos (KT), and the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

References to CLRS refer to the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein.

References to KT refer to the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos.

References to Rosen refer to the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

This document will inevitably contain mistakes. If you find some, please report them to your TA so that we can correct them.

Sheet 10

Rosen, 7.2, 6

Exercise

What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?

a) 1 precedes 3.

b) 3 precedes 1.

c) 3 precedes 1 and 3 precedes 2.

Rosen, 7.2, 11

Exercise

Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

Rosen, 7.2, 36

Exercise

Use mathematical induction to prove that if $E_1, E_2, \ldots, E_n$ is a sequence of n pairwise disjoint events in a sample space S, where n is a positive integer, then $p(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i)$.

Rosen, 7.2, 38

Exercise

A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

a. What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

b. Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?

Rosen, 7.4, 4

Exercise

A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times?

Rosen, 7.4, 8

Exercise

What is the expected sum of the numbers that appear when three fair dice are rolled?

Rosen, 7.4, 18

Exercise

Suppose that X and Y are random variables and that X and Y are nonnegative for all points in a sample space S. Let Z be the random variable defined by $Z(s) = max(X(s), Y(s))$ for all elements $s \in S$. Show that $\mathrm{E}(Z) \leq \mathrm{E}(X) + \mathrm{E}(Y)$.

Rosen, 7.4, 29.a

Exercise

Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

a) What is the expected value of X_n?

Rosen, 7.4, 37

Exercise

Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Show that $p(X(s) \geq a) \leq E(X)/a$ for every positive real number a. This inequality is called **Markov's inequality**.

Rosen, 7.4, 49

Exercise

What is the expected number of bins that remain empty when m balls are distributed into n bins uniformly at random?

Rosen, 7.4, supplementary exercise 13

Exercise

Suppose n people, $n \geq 3$, play "odd person out" to decide who will buy the next round of refreshments. The n people each flip a fair coin simultaneously. If all the coins but one come up the same, the person whose coin comes up different buys the refreshments. Otherwise, the people flip the coins again and continue until just one coin comes up different from all the others.

a) What is the probability that the odd person out is decided in just one coin flip?

b) What is the probability that the odd person out is decided with the kth flip?

c) What is the expected number of flips needed to decide the odd person out with n people?

Exercise from course webpage

Exercise

For a graph, $G = (V, E)$, a *spanning bipartite subgraph* G' of G is defined by a partition (V_1, V_2) of V, and the edges $E' \subseteq E$ that have an endpoint in both parts.

Consider the following randomized algorithm for finding a spanning bipartite subgraph of an arbitrary graph: Independently, for each vertex $v \in V$, decide uniformly at random if vertex v is in V_1 or V_2.

1. Give a lower bound on the expected number of edges m' in E' as a function of $m = |E|$.
2. How can you use your result to conclude that any graph has a spanning bipartite subgraph with $m' \geq m/2$?
3. Design a deterministic, polynomial-time algorithm for this problem, finding a spanning bipartite subgraph G' of any graph G, where $m' \geq m/2$.