Sidste gang: <u>Induktionsbeviser</u>

<u>Eks</u>: $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$2^0 = 1 = 2^1 - 1 \quad \checkmark$

$2^0 = 2^1 - 1 \quad \Rightarrow$

$2^0 + 2^1 = 2^1 + 2^1 - 1 = 2^2 - 1 \quad \Rightarrow$

$2^0 + 2^1 + 2^2 = 2^2 + 2^2 - 1 = 2^3 - 1 \quad \Rightarrow$

....

Generelt:

$2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1 \quad \Rightarrow$

$2^0 + 2^1 + \ldots + 2^{n-1} + 2^n = 2^n + 2^n - 1 = 2^{n+1} - 1$

Afsnit 2.1-2.2:

Mængder (uordnet samling af elementer)

Eks: $\underbrace{\mathbb{N}, \mathbb{R}, \mathbb{Q},}_{\text{uendelige mængder}} \underbrace{\{1,2,3\}, \text{ danske bogstaver}}_{\text{endelige mængder}}$

$$M = \{1,2,3\} = \{3,1,2\} = \{3,1,1,2,3\}$$

$$V = \{a, e, i, o, u, y\}$$ opremsning (enumeration)

$$S = \{1,2,3,\ldots,100\}$$

$$\mathbb{N} = \{0,1,2,\ldots\}$$ uendelig opremsning

$$\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$$

$$L = \{0,2,4,6,\ldots\}$$

$$= \{x \in \mathbb{N} \mid 2 \mid x\}$$ mængde-bygger-notation

„hvorom der gælder" „går op i" (Kap. 4)

$$= \{x \in \mathbb{N} \mid \exists k \in \mathbb{N} : x = 2k\}$$

$$= \{2x \mid x \in \mathbb{N}\}$$

Mængde-bygger-notation: $\{\langle\text{element}\rangle [\in \langle\text{univers}\rangle] \mid \langle\text{udsagn}\rangle\}$

$$\{1,2,4,8,\ldots\} = \{2^n \mid n \in \mathbb{N}\}$$

$$\{1,2,4,8,16\} = \{2^n \mid n \in \mathbb{N} \wedge n \leq 4\}$$

$$\mathbb{Q} = \left\{\frac{p}{q} \;\middle|\; p \in \mathbb{Z} \wedge q \in \mathbb{Z} - \{0\}\right\}$$ (rationale tal)

$$= \left\{\frac{p}{q} \;\middle|\; p \in \mathbb{Z} \wedge q \in \mathbb{Z}^+\right\}$$

$$\mathbb{Q}^+ = \left\{\frac{p}{q} \;\middle|\; p \in \mathbb{Z}^+ \wedge q \in \mathbb{Z}^+\right\}$$

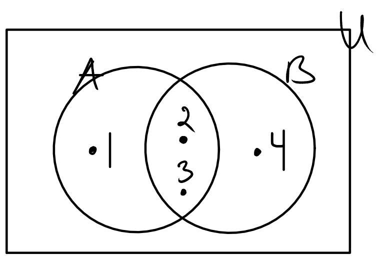
$|S|$: kardinaliteten af S („størrelsen" af S)
elementer i S, hvis S er endelig
Dat + Mat venter tillæge til kardinalitet af uendelige mængder

Eks: $|V| = 6$

$|S| = 100$

$|M| = |\{1,2,3\}| = |\{3,2,1\}| = |\{3,1,1,2,3\}| = 3$

Eks: $A = \{1,2,3\}$, $B = \{2,3,4\}$, $U = \mathbb{N}$



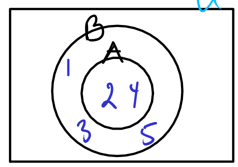
Venn-diagram

Delmængde (subset)

$A \subseteq B \iff \forall x \in U : (x \in A \Rightarrow x \in B) \iff \forall x \in A : x \in B$

„ er en delmængde af "
„ er indeholdt i "

U: universet



Ven-diagram

Eks:

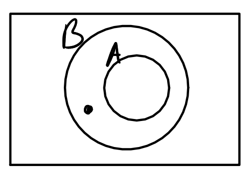
$A = \{2, 4\}, \quad B = \{1, 2, 3, 4, 5\}$

$\mathbb{N} \subseteq \mathbb{R}$

$\mathbb{R} \subseteq \mathbb{R}$

Ægte delmængde: (proper subset)

$A \subset B \iff A \subseteq B \wedge A \neq B \iff A \subseteq B \wedge B - A \neq \emptyset$



$\emptyset = \{\}$ Den tomme mængde

Sætning 2.1.1.

For enhver mængde S gælder

(i) $\emptyset \subseteq S$

(ii) $S \subseteq S$

Bevis:

(i) Skal vise:

$$\forall x \in U : \underbrace{(\underbrace{x \in \emptyset}_{F} \Rightarrow x \in S)}_{S} \quad \checkmark$$

(ii) Skal vise:

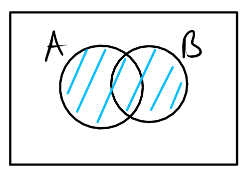
$$\forall x \in U : (x \in S \Rightarrow x \in S) \quad \checkmark$$

$\square$

Foreningsmængde (union)

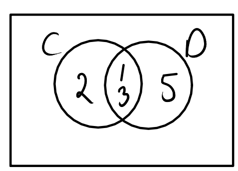
$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

$\uparrow$ „forenet med"



Eks: $C = \{1,2,3\}$, $D = \{1,3,5\}$

$C \cup D = \{1,2,3,5\}$



Eks: $\mathbb{N} \cup \mathbb{R} = \mathbb{R}$

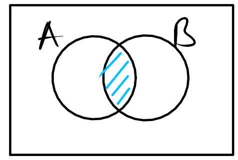
men $\{\mathbb{N}, \mathbb{R}\} \neq \mathbb{R}$

$|\{\mathbb{N}, \mathbb{R}\}| = 2$

Fællesmængde: (intersection)
(snitmængde)

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

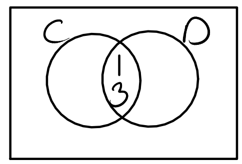
"fælles med"
"snit"



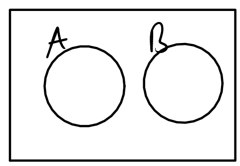
Eks:

$C = \{1,2,3\}$, $D = \{1,3,5\}$

$C \cap D = \{1,3\}$



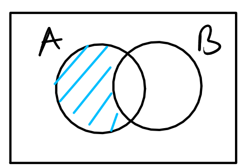
A og B er disjunkte, hvis $A \cap B = \emptyset$
(disjoint)



A fraregnet B (difference)

$A - B = \{x \in A \mid x \notin B\}$

$A \setminus B$

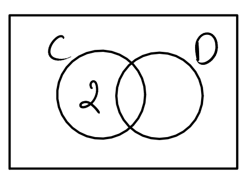


Eks:

$C = \{1,2,3\}$, $D = \{1,3,5\}$

$C - D = \{2\}$

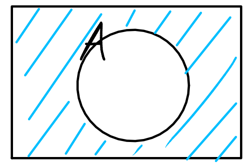
$D - C = \{5\}$



$A - B = A \iff A \cap B = \emptyset$ (dvs A og B er disjunkte)

Komplement

$$\overline{A} = \{x \in U \mid x \notin A\}$$
$$= U - A$$



Eks: $U = \mathbb{N}$, $L = \{0, 2, 4, \dots\}$

$\overline{L} = \{1, 3, 5, \dots\}$

$U = \{1, 2, 3, 4, 5\}$

$\overline{C} = \{4, 5\}$, $\overline{D} = \{2, 4\}$

Mængde-ligheder

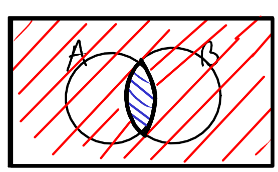
Tabel 2.2.1, s. 132 ~ Tabel 1.3.6 s. 25

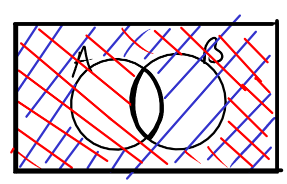
$\cup$ ~ $\vee$

$\cap$ ~ $\wedge$

De Morgans Love

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$





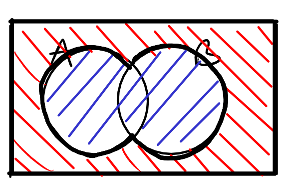
: $A \cap B$

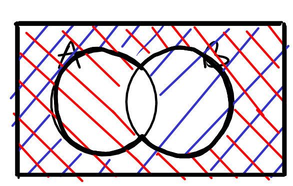
: $\overline{A \cap B}$

: $\overline{A}$: $\overline{B}$

, og : $\overline{A} \cup \overline{B}$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$





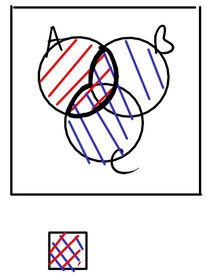
: $\overline{A \cup B}$

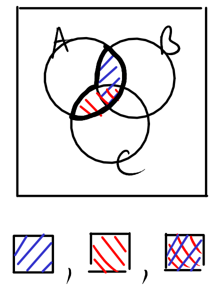
: $\overline{A} \cap \overline{B}$

Bogen beviser love fra Tabel 22.1 v.h.a. omskrivninger af def. af de enkelte mængde-operationer og v.h.a. tilhørs-tabeller (membership tables), men det er som regel nemmest at bruge Venn-diagrammer (som ovenfor).

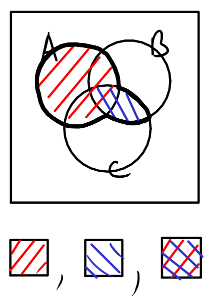
Distributive love:

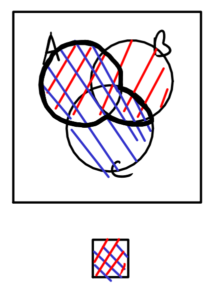
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$





$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$





Potensmængde (power set)

$P(S) = \{ A \mid A \subseteq S \} = 2^S$

Eks:

$S = \emptyset$

$P(S) = \{\emptyset\}$ $\qquad |P(S)| = 1 = 2^0$

$S = \{1\}$

$P(S) = \{\emptyset, \{1\}\}$ $\qquad |P(S)| = 2 = 2^1$

$S = \{1,2\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ $\qquad |P(S)| = 4 = 2^2$

Påstand: En mængde S med n elementer har 2^n delmængder

Bevis: Ved induktion over n

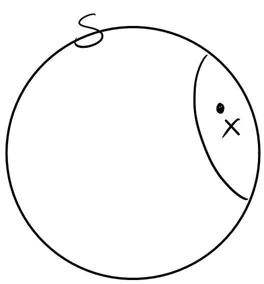
Basis: $n=0$

Den tomme mængde har én delmængde, nemlig $\emptyset$.

Ind. ant.:

En mængde med $n-1$ elementer har 2^{n-1} delmængder

Ind. skridt: $n \geq 1$



$|S| = n$

x: vilkårligt element i S

$S' = S - \{x\}$, dvs. $|S'| = n-1$.

S' har 2^{n-1} delmængder iflg. ind. ant.

For enhver delmængde $A \subseteq S$ er der to muligheder:

- $x \notin A$, dvs. $A \subseteq S'$ (2^{n-1} muligheder)
- $x \in A$, dvs. $A = \{x\} \cup A'$, hvor $A' \subseteq S'$ (2^{n-1} muligheder)

I alt $2^{n-1} + 2^{n-1} = 2^n$ muligheder $\square$

Kartesisk produkt

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

$$|A \times B| = |A| \cdot |B|$$

Eks: $A = \{1,2\}, \quad B = \{2,3\}$

$$A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$$

Lidt notation:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

(ligegyldigt, hvordan man sætter parenteser)