DM826 – Spring 2011

Modeling and Solving Constrained Optimization Problems

Lecture 12

Column Generation and Branch and Price

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Resume

- Modelling in IP and CP
- Global constraints
- Local consistency notions
- Filtering algorithms for global constraints
- Search
- Symmetries
- Set variables
- Integrated/Advanced Approaches:
    - Branch and price
    - Logic-based Benders decomposition
- Scheduling

Tractable Structures

Binary constraints $\rightsquigarrow$ constraint graph

- Independent subproblems
  connected components
- Tree-networks
  directed arc coinsistency enforced in linear time
- Reduce graphs to trees
  - cutset conditioning (cycle cutset, NP-hard problem)
  - tree decomposition (min tree width NP-hard problem)

Possible extensions to hypergraphs

Outline

Hybridization schemes

CP and IP hybridization schemes:

- relaxations
  (eg, bound filterning in linear constraints and guiding search in soft constraints)
- decomposition approaches:
  - Branch and price
  - Benders-based decompiosition

Dantzig-Wolfe Decomposition

Motivation: Large difficult IP models

➨ split them up into smaller pieces

Applications

- Cutting Stock problems
- Multicommodity Flow problems
- Facility Location problems
- Capacitated Multi-item Lot-sizing problem
- Air-crew and Manpower Scheduling
- Vehicle Routing Problems
- Scheduling

Leads to methods also known as:

- Branch-and-price (column generation + branch and bound)
- Branch-price-and-cut (column generation + branch and bound + cutting planes)

Dantzig-Wolfe Decomposition

The problem is split into a master problem and a subproblem

+ Tighter bounds
+ Better control of subproblem
− Model may become (very) large

Delayed column generation

Write up the decomposed model gradually as needed

- Generate a few solutions to the subproblems
- Solve the master problem to LP-optimality
- Use the dual information to find most promising solutions to the subproblem
- Extend the master problem with the new subproblem solutions.

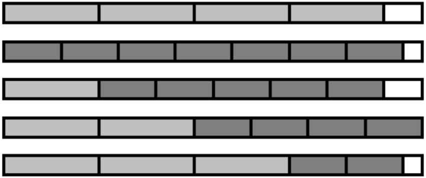
Motivation: Cutting stock problem

- Infinite number of raw stocks, having length L.
- Cut m piece types i, each having width w_i and demand b_i.
- Satisfy demands using least possible raw stocks.

Example:

- $w_1 = 5, b_1 = 7$ 
- $w_2 = 3, b_2 = 3$ 
- Raw length $L = 22$

Some possible cuts



Formulation 1

$$
\begin{array}{ll}
\text{minimize} & u_1 + u_2 + u_3 + u_4 + u_5 \\
\text{subject to} & 5x_{11} + 3x_{12} \leq 22u_1 \\
& 5x_{21} + 3x_{22} \leq 22u_2 \\
& 5x_{31} + 3x_{32} \leq 22u_3 \\
& 5x_{41} + 3x_{42} \leq 22u_4 \\
& 5x_{51} + 3x_{52} \leq 22u_5 \\
& x_{11} + x_{21} + x_{31} + x_{41} + x_{51} \geq 7 \\
& x_{12} + x_{22} + x_{32} + x_{42} + x_{52} \geq 3 \\
& u_j \in \{0, 1\} \\
& x_{ij} \in \mathbb{Z}_+
\end{array}
$$

LP-relaxation gives solution value $z = 2$ with

$$u_1 = u_2 = 1, x_{11} = 2.6, x_{12} = 3, x_{21} = 4.4$$

Block structure

$$
\begin{array}{lccccccccccc}
\text{min} & & u_1 & & +u_2 & & +u_3 & & +u_4 & & +u_5 & \\
\text{s.t} & x_{11} & & +x_{21} & & +x_{31} & & +x_{41} & & +x_{51} & & \geq 7 \\
& x_{12} & & +x_{22} & & +x_{32} & & +x_{42} & & +x_{51} & & \geq 3 \\
& \multicolumn{2}{c}{5x_{11} + 3x_{12} - 22u_1} & & & & & & & & & \leq 0 \\
& & & \multicolumn{2}{c}{5x_{21} + 3x_{22} - 22u_2} & & & & & & & \leq 0 \\
& & & & & \multicolumn{2}{c}{5x_{31} + 3x_{32} - 22u_3} & & & & & \leq 0 \\
& & & & & & & \multicolumn{2}{c}{5x_{41} + 3x_{42} - 22u_4} & & & \leq 0 \\
& & & & & & & & & \multicolumn{2}{c}{5x_{51} + 3x_{52} - 22u_5} & \leq 0
\end{array}
$$

Formulation 2

The matrix A contains all different cutting patterns
All (undominated) patterns:

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$

Problem

$$\begin{aligned} \text{minimize } & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to } & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{aligned}$$

LP-relaxation gives solution value $z = 2.125$ with

$$\lambda_1 = 1.375, \lambda_4 = 0.75$$

Due to integer property a lower bound is $\lceil 2.125 \rceil = 3$.
Optimal solution value is $z^* = 3$.

Round up LP-solution getting heuristic solution $z_H = 3$.

Decomposition

If model has "block" structure

$$\begin{array}{lcccccccl}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = & b \\
& D^1x^1 & + & & & & \leq & d_1 \\
& & + & D^2x^2 & & & \leq & d_2 \\
& & & & \ldots & & \leq & \vdots \\
& & & & & D^Kx^K & \leq & d_K \\
& x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

Lagrangian relaxation

Objective becomes

$$\begin{array}{l}
c^1x^1 + c^2x^2 + \ldots + c^Kx^K \\
-\lambda\left(A^1x^1 + A^2x^2 + \ldots + A^Kx^K - b\right)
\end{array}$$

Decomposed into

$$\begin{array}{lcccccl}
\max c^1x^1 - \lambda A^1x^1 + c^2x^2 - \lambda A^2x^2 + \ldots + c^Kx^K - \lambda A^Kx^K + \; b & & & & & & \\
\text{s.t.} \quad D^1x^1 & + & & & & \leq & d_1 \\
& + & D^2x^2 & & & \leq & d_2 \\
& & & \ldots & & \leq & \vdots \\
& & & & D^Kx^K & \leq & d_K \\
x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

Model is separable

Dantzig-Wolfe decomposition

If model has "block" structure

$$\begin{array}{llllll} \max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = b \\ & D^1x^1 & + & & & & \leq d_1 \\ & & + & D^2x^2 & & & \leq d_2 \\ & & & & \ldots & & \leq \vdots \\ & & & & & D^Kx^K & \leq d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & \end{array}$$

Describe each set X^k, $k = 1, \ldots, K$

$$\begin{array}{llllll} \max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = b \\ & x^1 \in X^1 & & x^2 \in X^2 & \ldots & x^K \in X^K & \end{array}$$

where $X^k = \{x^k \in \mathbb{Z}_+^{n_k} : D^k x^k \leq d_k\}$

Assuming that X^k has finite number of points $\{x^{k,t}\}$ $t \in T_k$

$$X^k = \left\{ \begin{array}{ll} x^k \in \mathbb{R}^{n_k} : & x^k = \sum_{t \in T_k} \lambda_{k,t} x^{k,t}, \\ & \sum_{t \in T_k} \lambda_{k,t} = 1, \\ & \lambda_{k,t} \in \{0,1\}, t \in T_k \end{array} \right\}$$

Dantzig-Wolfe decomposition

Substituting X^k in original model getting *Master Problem*

$$\max c^1(\sum_{t\in T_1}\lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2}\lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K}\lambda_{K,t}x^{K,t})$$

$$\text{s.t.}\quad A^1(\sum_{t\in T_1}\lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2}\lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K}\lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k}\lambda_{k,t} = 1 \qquad\qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \in \{0,1\}, \qquad\qquad t\in T_k \;\; k = 1,\ldots,K$$

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$
\begin{array}{llllll}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^kx^k & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^kx^k & = b \\
 & x^1 \in \text{conv}(X^1) & & x^2 \in \text{conv}(X^2) & \ldots & x^k \in \text{conv}(X^k) &
\end{array}
$$

Proof: Consider LP-relaxation

$$\max c^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t})$$

$$\text{s.t.} \quad A^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k} \lambda_{k,t} = 1 \qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \geq 0, \qquad t \in T_k \qquad k = 1,\ldots,K$$

Informally speaking we have

- joint constraint is solved to LP-optimality
- block constraints are solved to IP-optimality

Strength of Lagrangian relaxation

- z^{LPM} be LP-solution value of master problem
- z^{LD} be solution value of lagrangian dual problem

(Theorem 11.2)

$$z^{LPM} = z^{LD}$$

Proof: Lagrangian relaxing joint constraint in

$$\begin{array}{llllllll}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = & b \\
 & D^1x^1 & + & & & & \leq & d_1 \\
 & & + & D^2x^2 & & & \leq & d_2 \\
 & & & & \ldots & & \leq & \vdots \\
 & & & & & D^Kx^K & \leq & d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

Using result next page

$$\begin{array}{llllllll}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^kx^k & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^kx^k & = & b \\
 & x^1 \in \text{conv}(X^1) & & x^2 \in \text{conv}(X^2) & \ldots & x^k \in \text{conv}(X^k) & &
\end{array}$$

Strength of Lagrangian Relaxation (section 10.2)

Integer Programming Problem

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \le b \\ & Dx \le d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{aligned}$$

Lagrange Relaxation, multipliers $\lambda \ge 0$

$$\begin{aligned} \text{maximize} \quad & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} \quad & Ax \le b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{aligned}$$

for best multiplier $\lambda \ge 0$

$$\boxed{\max\left\{cx : Dx \le d, x \in \text{conv}(Ax \le b, x \in \mathbb{Z}_+)\right\}}$$

Delayed Column Generation

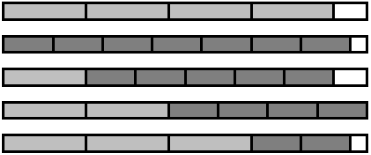
Delayed column generation, linear master

- Master problem can (and will) contain many columns
- To find bound, solve LP-relaxation of master
- Delayed column generation gradually writes up master

Delayed column generation, linear master

- $w_1 = 5$, $b_1 = 7$ 
- $w_2 = 3$, $b_2 = 3$ 
- Raw length $L = 22$

Some possible cuts



In matrix form

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 & \cdots \\ 0 & 7 & 5 & 4 & 2 & \cdots \end{pmatrix}$$

LP-problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

where

- $b = (7, 3)$,
- $x = (x_1, x_2, x_3, x_4, x_5, \cdots)$
- $c = (1, 1, 1, 1, 1, \cdots)$.

Reduced Costs

Simplex in matrix form

$$\min\{cx \mid Ax = b, x \geq 0\}$$

In matrix form:

$$\begin{bmatrix} A & 0 \\ c & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- $\mathcal{B} = \{1, 2, \ldots, p\}$ basic variables
- $\mathcal{L} = \{1, 2, \ldots, q\}$ non-basis variables (will be set to lower bound = 0)
- $(\mathcal{B}, \mathcal{L})$ basis structure
- $x_\mathcal{B}, x_\mathcal{L}, c_\mathcal{B}, c_\mathcal{L}$
- $B = [A_1, A_2, \ldots, A_p]$, $L = [A_{p+1}, A_{p+2}, \ldots, A_{p+q}]$

$$\begin{bmatrix} B & L & 0 \\ c_\mathcal{B} & c_\mathcal{L} & -1 \end{bmatrix} \begin{bmatrix} x_\mathcal{B} \\ x_\mathcal{L} \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$Bx_{\mathcal{B}} + Lx_{\mathcal{L}} = b \quad \Rightarrow \quad x_{\mathcal{B}} + B^{-1}Lx_{\mathcal{L}} = B^{-1}b \quad \Rightarrow \quad \left[\begin{array}{l} x_{\mathcal{L}} = 0 \\ x_{\mathcal{B}} = B^{-1}b \end{array} \right.$$

Simplex algorithm sets $x_{\mathcal{L}} = 0$ and $x_{\mathcal{B}} = B^{-1}b$ (for Fundamental Theorem)
B invertible, hence rows linearly independent

The objective function is obtained by multiplying and subtracting constraints by means of multipliers π (the dual variables)

$$z = \sum_{j=1}^{p} \left[c_j - \sum_{i=1}^{p} \pi_i a_{ij} \right] x_j + \sum_{j=p+1}^{p+q} \left[c_j - \sum_{i=1}^{p} \pi_i a_{ij} \right] x_j + \sum_{i=1}^{p} \pi_i b_i$$

Each basic variable has cost null in the objective function

$$c_j - \sum_{i=1}^{p} \pi_i a_{ij} = 0 \qquad \Longrightarrow \qquad \pi = B^{-1} c_{\mathcal{B}}$$

Reduced costs of non-basic variables:

$$c_j - \sum_{i=1}^{p} \pi_i a_{ij}$$

Delayed column generation (example)

- $w_1 = 5, b_1 = 7$ 
- $w_2 = 3, b_2 = 3$ 
- Raw length $L = 22$

Initially we choose only the trivial cutting patterns

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$$

Solve LP-problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

i.e.

$$\begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

with solution $x_1 = \frac{7}{4}$ and $x_2 = \frac{3}{7}$.
The dual variables are $y = c_B A_B^{-1}$ i.e.

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{7} \end{pmatrix}$$

Small example (continued)

Find entering variable

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 5 & 4 & 2 & \cdots \end{pmatrix} \qquad \begin{matrix} \frac{1}{4} \leftarrow y_1 \\ \frac{1}{7} \leftarrow y_2 \end{matrix}$$

$$c_N - yA_N = \left(1 - \tfrac{27}{28} \;\; 1 - \tfrac{30}{28} \;\; 1 - \tfrac{29}{28} \;\; \cdots \right)$$

We could also solve optimization problem

$$\begin{aligned} \min \;& 1 - \frac{1}{4}x_1 - \frac{1}{7}x_2 \\ \text{s.t.} \;& 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{aligned}$$

which is equivalent to knapsack problem

$$\begin{aligned} \max \;& \frac{1}{4}x_1 + \frac{1}{7}x_2 \\ \text{s.t.} \;& 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{aligned}$$

This problem has optimal solution $x_1 = 2, x_2 = 4$.
Reduced cost of entering variable

$$1 - 2\frac{1}{4} - 4\frac{1}{7} = 1 - \frac{30}{28} = -\frac{1}{14} < 0$$

Small example (continued)

Add new cutting pattern to A getting

$$A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 7 & 2 \end{pmatrix}$$

Solve problem to LP-optimality, getting primal solution

$$x_1 = \frac{5}{8}, x_3 = \frac{3}{2}$$

and dual variables

$$y_1 = \frac{1}{4}, y_2 = \frac{1}{8}$$

Note, we do not need to care about “leaving variable”
To find entering variable, solve

$$\begin{aligned} \max\ & \frac{1}{4}x_1 + \frac{1}{8}x_2 \\ \text{s.t.}\ & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{aligned}$$

This problem has optimal solution $x_1 = 4$, $x_2 = 0$.
Reduced cost of entering variable

$$1 - 4\frac{1}{4} - 0\frac{1}{7} = 0$$

Terminate with $x_1 = \frac{5}{8}$, $x_3 = \frac{3}{2}$, and $z_{LP} = \frac{17}{8} = 2.125$.

Questions (same as for the simplex method)

- Will the process terminate?

  Always improving objective value. Only a finite number of basis solutions.

- Can we repeat the same pattern?

  No, since the objective function is improved. We know the best solution among existing columns. If we generate an already existing column, then we will not improve the objective.

Branch-and-Price
Terminology

- Master Problem
- Restricted Master Problem
- Subproblem or Pricing Problem
- Branch and cut:
  Branch-and-bound algorithm using cuts to strengthen bounds.
- Branch and price:
  Branch-and-bound algorithm using column generation to derive bounds.

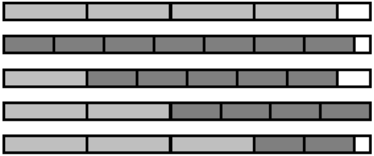
Branch-and-price

- LP-solution of master problem may have fractional solutions
- Branch-and-bound for getting IP-solution
- In each node solve LP-relaxation of master
- Subproblem may change when we add constraints to master problem
- Branching strategy should make subproblem easy to solve

Branch-and-price, example

The matrix A contains all different cutting patterns

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$



Problem

$$\begin{aligned} \text{minimize} \quad & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to} \quad & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{aligned}$$

LP-solution $\lambda_1 = 1.375, \lambda_4 = 0.75$

Branch on $\lambda_1 = 0$, $\lambda_1 = 1$, $\lambda_1 = 2$

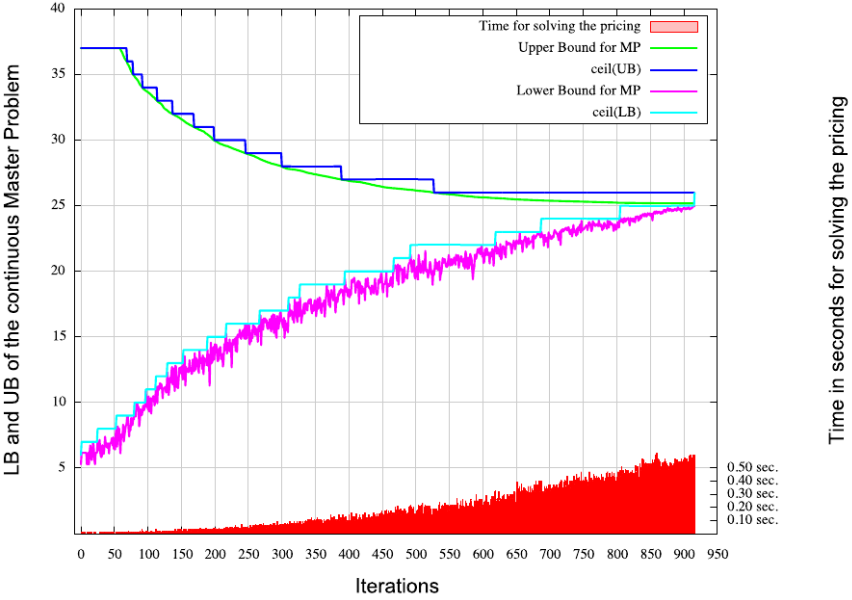
- Column generation may not generate pattern (4,0)
- Pricing problem is knapsack problem with pattern forbidden

Tailing off effect

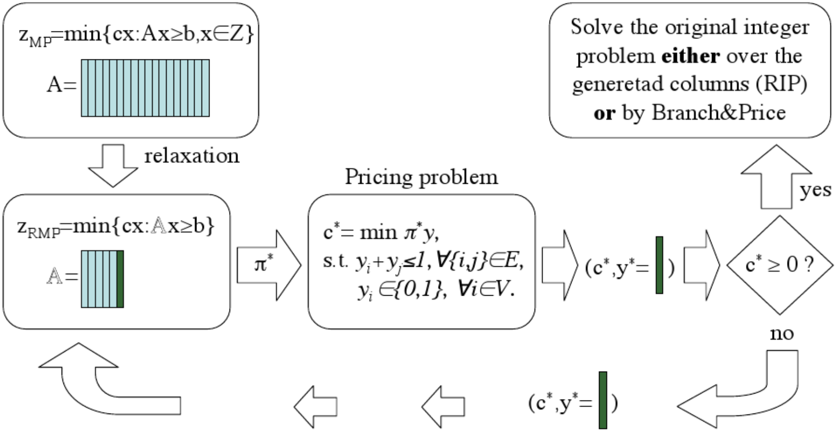
Column generation may converge slowly in the end

- We do not need exact solution, just lower bound
- Solving master problem for subset of columns does not give valid lower bound (why?)
- Instead we may use Lagrangian relaxation of joint constraint
- "guess" Lagrangian multipliers equal to dual variables from master problem

Convergence in CG



[plot by Stefano Gualandi, Politecnico di Milano]



[illustration by Stefano Gualandi, Politecnico di Milano]
(the pricing problem is for a GCP)

Heuristic solution (eg, in sec. 12.6)

- Restricted master problem will only contain a subset of the columns
- We may solve restricted master problem to IP-optimality
- Restricted master is a "set-covering-like" problem which is not too difficult to solve

References

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