DM826 – Spring 2012

Modeling and Solving Constrained Optimization Problems

Lecture 5

Constraint Propagation and Local Consistency

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Reasoning with Constraints

Constraint Propagation, aka:

- constraint relaxation
- filtering algorithms
- narrowing algorithms
- constraint inference
- simplification algorithms
- label inference
- local consistency enforcing
- rules iteration
- chaotic iteration

Local Consistency define properties that the constraint problem must satisfy *after* constraint propagation

Rules iteration defines properties on the process of propagation itself, that is is kind and order of operations of reduction applied to the problem

Notation and Terminology

Finite domains $\rightsquigarrow$ w.l.g. $D \subseteq \mathbf{Z}$

Constraint C: relation on a (ordered) *subsequence* of variables

- $X(C) = (x_{i_1}, \ldots, x_{i_{|X(C)|}})$ is the scheme or scope
- $|X(C)|$ is the arity of C (unary/binary/non-binary)
- $C \subseteq \mathbf{Z}^{|X(C)|}$ containing combinations of valid values (or tuples)
  $\tau \in \mathbf{Z}^{|X(C)|}$
- constraint check: testing whether a τ satisfies C
- $\mathcal{C}$: a t-tuple of constraints $\mathcal{C} = (C_1, \ldots, C_t)$
- expression
  - extensional: specifies satisfying tuples (aka `table` in Comet, `extensional` via `DFA` or `TupleSet` in gecode).
    eg. $c(x_1, x_2) = \{(2,2), (2,3), (3,2), (3,3)\}$
  - intensional: specifies the characteristic function. eg. $\texttt{alldiff}(x_1, x_2, x_3)$

CSP

Input:

- **Variables** $X = (x_1, \ldots, x_n)$
- **Domain Expression** $\mathcal{DE} = \{x_1 \in D(x_1), \ldots, x_n \in D(x_n)\}$

a constrained satisfaction problem (CSP) is

$$\mathcal{P} = \langle X, \mathcal{DE}, \mathcal{C} \rangle$$

$\mathcal{C}$ finite set of constraints each on a subsequence of X.
$C \in \mathcal{C}$ on $Y = (y_1, \ldots, y_k)$ is $C \subseteq D(y_1) \times \ldots \times D(y_k)$

$(v_1, \ldots, v_n) \in D(x_1) \times \ldots \times D(x_n)$ is a solution of $\mathcal{P}$
if for each constraint $C_i \in \mathcal{C}$ on $x_{i_1} \ldots, x_{i_m}$ it is

$$(v_{i_1}, \ldots, v_{i_m}) \in C_i$$

CSP normalized: iff two different constraints do not involve exactly the same vars
CSP binary iff for all $C_i \in \mathcal{C}, |X(C)| = 2$

Notation and Terminology

Given a tuple τ on a sequence Y of variables and $W \subseteq Y$,

- $\tau[W]$ is the restriction of τ to variables in W (ordered accordingly)
- $\tau[x_i]$ is the value of x_i in τ
- if $X(C) = X(C')$ and $C \subseteq C'$ then for all $\tau \in C$ the reordering of τ according to $X(C')$ satisfies C'.

Example

$C(x_1, x_2, x_3)$: $\quad x_1 + x_2 = x_3$
$C'(x_1, x_2, x_3)$: $\quad x_1 + x_2 \leq x_3$

$C \subseteq C'$

Notation and Terminology

- Given $Y \subseteq X(C)$, $\pi_Y(C)$ denotes the projection of C on Y. It contains tuples on Y that can be extended to a tuple on $X(C)$ satisfying C.
- given $X(C_1) = X(C_2)$, $C_1 \cup C_2$ contains the tuples τ that satisfy both c_1 and c_2
- join of $\{C_1 \dots C_k\}$ is the relation with scheme $\cup_{i=1}^{k} X(C_i)$ that contains tuples such that $\tau[X(c_i)] \in c_i$ for all $1 \leq i \leq k$.

Notation and Terminology

Given $\mathcal{P} = \langle X, \mathcal{DE}, \mathcal{C} \rangle$ the instantiation I is a tuple on $Y = (x_1, \ldots, x_k) \subseteq X$: $((x_1, v_1), \ldots, (x_k, v_k))$

- I on Y is valid iff $\forall x_i \in Y$, $I[x_i] \in D(x_i)$
- I on Y is locally consistent on Y iff it is valid and for all $C \in \mathcal{C}$ with $X(C) \subseteq Y$, $I[X(C)]$ satisfies C
- a solution to $\mathcal{P}$ is an instantiation I on $X(C)$ which is locally consistent
- I on Y is globally consistent if it can be extended to a solution, i.e., there exists $s \in sol(\mathcal{P})$ with $I = s[Y]$

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_i) = \{1..5\}, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv \texttt{alldiff}(x_1, x_2, x_3), C_2 \equiv x_1 \leq x_2 \leq x_3, C_3 \equiv x_4 \geq 2x_2\}\rangle$$

$\pi_{x_1, x_2}(C_1) \equiv (x_1 \neq x_2)$
$I_1 = ((x_1, 1), (x_2, 2), (x_4, 7))$ is not valid
$I_2 = ((x_1, 1), (x_2, 1), (x_4, 3))$ is local consistent since C_3 only one with $X(C_3) \subseteq Y$ and $I_2[X(C_3)]$ satisfies C_3
I_2 is not global consistent: $sol(\mathcal{P}) = \{(1, 2, 3, 4), (1, 2, 3, 5)\}$

Notation and Terminology

CSP is NP-complete!
$\rightsquigarrow$ solved by extending partial instantiations to global consistent ones

$\mathcal{P}$' is a tightening of $\mathcal{P}$ if
$X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{DE}_{\mathcal{P}'} \subseteq \mathcal{DE}, \quad \forall C \in \mathcal{C}, \exists C' \in \mathcal{C}, X(C') = X(C)$ and $C' \subseteq C$.
That is, any instantiation I on $Y \subseteq X_{\mathcal{P}}$ locally inconsistent for $\mathcal{P}$ is locally inconsistent for $\mathcal{P}'$.

$\mathcal{S}_{\mathcal{P}}$ is the space of all tightening for $\mathcal{P}$

We are interested in the tightenings that preserve the set of solutions ($\mathrm{sol}(\mathcal{P}') = \mathrm{sol}(\mathcal{P})$) whose space is denoted $\mathcal{S}_{\mathcal{P}}^{\mathrm{sol}}$ and among them the smallest

$\mathcal{P}^* \in \mathcal{S}_{\mathcal{P}}^{\mathrm{sol}}$ is global consistent if any instantiation I on $Y \subseteq X$ which is locally consistent in $\mathcal{P}^*$ can be extended to a solution of $\mathcal{P}$.

Computing $\mathcal{P}^*$ is exponential in time and space $\rightsquigarrow$ search a close $\mathcal{P}$ in polynomial time and space

Define a property Φ that states necessary conditions on instantiations that enter in the definition of local consistency

Constraint Propagation

We reach a $\mathcal{P}$ that is Φ consistent by constraint propagation:

- tighten $\mathcal{DE}$
- tighten $\mathcal{C}$, ex: $x_1 + x_2 \leq x_3 \rightsquigarrow x_1 + x_2 = x_3$
- add C to $\mathcal{C}$

this is implemented by

- reduction rules: sufficient conditions to rule out values that have no chance to appear in a solution (defined through local consistency property Φ).
- rules iterations: a set of reduction rules for each set of constraint that tighten the problem

Focus on domain-based tightenings

Domain-based tightenings

The space $\mathcal{S}_\mathcal{P}$ of domain-based tightenings of $\mathcal{P}$ is the set of problems $\mathcal{P}' = \langle X', \mathcal{DE}', \mathcal{C}' \rangle$ such that $X_{\mathcal{P}'} = X_\mathcal{P}, \quad \mathcal{DE}_{\mathcal{P}'} \subseteq \mathcal{DE}, \quad \mathcal{C}' = \mathcal{C}$

Task:
Finding a tightening $\mathcal{P}^*$ in $\mathcal{S}_\mathcal{P}^{\text{sol}} \subseteq \mathcal{S}_\mathcal{P}$ (the set that contains all problems that preserve the solutions of $\mathcal{P}$) such that:
forall $x_i \in X_\mathcal{P}$, $D_{\mathcal{P}*}(x_i)$ contains only values that belong to a solution itself, i.e., $D_{\mathcal{P}*}(x_i) = \pi_{\{x_i\}}(\text{sol}(\mathcal{P}))$

It is clearly NP-hard since it corresponds to solving $\mathcal{P}$ itself.

- Reduction rules:

  $D(x_i) \leftarrow D(x_i) \cap \{v_i | D(x_1) \times D(x_j - 1) \times \{v_i\} \times \ldots D(x_j + 1) \times \ldots D(x_k) \cap C \neq \emptyset\}$

- Rules iterations

Define Φ: e.g., unary, arc, path, k-consistency

Domain-based local consistency

Domain-based local consistency property Φ specifies a necessary condition on values to belong to solutions

A domain-based property Φ is stable under union iff for any Φ-consistent problem $\mathcal{P}_1 = (X, \mathcal{DE}, \mathcal{C})$ and $\mathcal{P}_2 = (X, \mathcal{DE}, \mathcal{C})$ the problem $\mathcal{P}' = (X, \mathcal{DE}_1 \cup \mathcal{DE}_2, C)$ is Φ-consistent.

Example

Φ for each constraint C and variable $x_i \in X(C)$, at least half of the values in $D(x_i)$ belong to a valid tuple satisfying C.

$$\mathcal{P} = \langle X = (x_1, x_2), \mathcal{DE} = \{D_1(x_1) = \{1, 2\}, D_1(x_2) = \{2\}\}, C \equiv \{x_1 = x_2\}\rangle$$
$$\mathcal{P} = \langle X = (x_1, x_2), \mathcal{DE} = \{D_2(x_1) = \{2, 3\}, D_2(x_2) = \{2\}\}, C \equiv \{x_1 = x_2\}\rangle$$

Both are Φ consistent but they are not stable under union.

Domain-based tightenings

Note: Not all Φ-consistent tightenings preserve the solutions
We search for the Φ-closure $\Phi(\mathcal{P})$ (the union of all $\mathcal{P}' \in \mathcal{S}_{\mathcal{P}}$ Φ-consistent)

$\equiv$ enforcing Φ consistency

$\mathrm{sol}(\phi(\mathcal{P})) = \mathrm{sol}(\mathcal{P})$

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_i) = \{1, 2\}, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv x_1 \leq x_2, C_2 \equiv x_2 \leq x_3, C_3 \equiv x_1 \neq x_3\}\rangle$$

Φ all values for all variables can be extended consistently to a second variable

$$\mathcal{P}' = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_1) = 1, D(x_2) = 1, D(x_3) = 2, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv x_1 \leq x_2, C_2 \equiv x_2 \leq x_3, C_3 \equiv x_1 \neq x_3\}\rangle$$

$\mathcal{P}$' is consistent but it does not contain $(1, 2, 2)$ which is in $\mathrm{sol}(\mathcal{P})$
$\Phi(\mathcal{P}) : \langle X, \mathcal{DE}_\Phi, \mathcal{C}\rangle$ with $D_\Phi(x_1) = 1, D_\Phi(x_2) = \{1, 2\}, D_\Phi(x_3) = 2$

Domain-based tightenings

Proposition (Fixed Point): If a domain based consistency property Φ is stable under union, then for any $\mathcal{P}$, the $\mathcal{P}$' with $\mathcal{DE}_{\mathcal{P}'}$ obtained by iteratively removing values that do not satisfy Φ until no such value exists is the Φ-closure of $\mathcal{P}$.

Contrary to $\mathcal{P}^*$, $\Phi(\mathcal{P})$ can be computed by a greedy algorithm:

Corollary If a domain-based consistency property Φ is polynomial to check, finding $\Phi(\mathcal{P})$ is polynomial as well.

enforcing Φ consistency $\equiv$ finding closure $\Phi(\mathcal{P})$

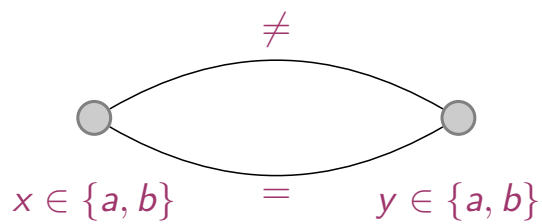
Possible to define a partial order
(For a, b, elements of a poset P, if $a \leq b$ or $b \leq a$, then a and b are comparable. Otherwise they are incomparable)
That is, Φ_1 is at least as strong as another Φ_2 if for any $\mathcal{P}$: $\Phi_1(\mathcal{P}) \leq \Phi_2(\mathcal{P})$,
ie, $X_{\Phi_1(\mathcal{P})} = X_{\Phi_2(\mathcal{P})}$, $\mathcal{DE}_{\Phi_1(\mathcal{P})} \subseteq \mathcal{DE}_{\Phi_2(\mathcal{P})}$, $\mathcal{C}_{\Phi_1(\mathcal{P})} = \mathcal{C}_{\Phi_2(\mathcal{P})}$
(any instantiation I on $Y \subseteq X_{\Phi_2(\mathcal{P})}$ locally inconsistent in $\Phi_2(\mathcal{P})$ is locally inconsistent in $\Phi_1(\mathcal{P})$)

Arc Consistency

(On binary CSP)
Arc consistency: every value in a domain is consistent with every constraint.

- $C = c(x, y)$ with $\mathcal{DE} = \{D(x), D(y)\}$ is arc consistent iff
  - $\forall a \in D(x)$ there exists $b \in D(y)$ such that $(a, b) \in C$
  - $\forall b \in D(y)$ there exists $a \in D(x)$ such that $(a, b) \in C$
- $\mathcal{P}$ is arc consistent iff it is AC for all its binary constraints

In general arc consistency does not imply global consistency.
An arc consistent but inconsistent CSP:



A consistent but not arc consistent CSP:

Generalized Arc Consistency (GAC)

Given arbitrary (non-normalized, non-binary) $\mathcal{P}$, $C \in \mathcal{C}$, $x_i \in X(C)$

(Value) $v \in D(x_i)$ is consistent with C in $\mathcal{DE}$ iff $\exists$ a valid tuple τ for C: $v_i = \tau[x_i]$. τ is called support for (x_i, v_i)

(Variable) $\mathcal{DE}$ is GAC on C for x_i iff all values in $D(x_i)$ are consistent with C in $\mathcal{DE}$ (i.e., $D(x_i) \subseteq \pi_{\{x_i\}}(C \cap \pi_{\{X(C)\}}(\mathcal{DE}))$)

(Problem) $\mathcal{P}$ is GAC iff $\mathcal{DE}$ is GAC for all v in X on all $C \in \mathcal{C}$

$\mathcal{P}$ is arc inconsistent iff the only domain tighter than $\mathcal{DE}$ which is GAC for all variables on all constraints is the empty set.

(aka, hyperarc consistency, domain consistency)
Example: arc consistency $\neq$ 2-consistency, AC $<$ 2C on non-normalized binary CSP, and incomparable on arbitrary CSP

References

Bessiere C. (2006). **Constraint propagation**. In *Handbook of Constraint Programming*, edited by F. Rossi, P. van Beek, and T. Walsh, chap. 3. Elsevier. Also as Technical Report LIRMM 06020, March 2006.