DM545

Linear and Integer Programming

Lecture 1
Introduction

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Outline

Outline

Context

Students:

- Computer Science (3rd year)
- Applied Mathematics (3rd year)
- Math-economy (3rd year)

Prerequisites

- Calculus (MM501, MM502)
- Linear Algebra (MM505)

Practical Information

Teacher: Marco Chiarandini (marco@imada.sdu.dk)
Instructor: Sushmita Gupta (sushmita.gupta@gmail.com)

Schedule ($\approx$ 24 lecture hours + $\approx$ 20 exercise hours):

Week	15	16	17	18	19	20	21
Tir, 14-16	Forelæsning (U20)	Forelæsning (U20)	Forelæsning (U20)	Forelæsning (U20)	Forelæsning (U20)	Forelæsning (U20)	
Ons, 08-10		Eksaminatorie (U140)	Eksaminatorie (U140)	Eksaminatorie (U140)	Eksaminatorie (U140)	Eksaminatorie (U140)	Eksaminatorie (U140)
Tor, 14-16	Forelæsning (U20)	Forelæsning (U20)	Forelæsning (U155)	Forelæsning (U20)	Forelæsning (U20)		
Fre, 12-14		Eksaminatorie (U20)	Eksaminatorie (U20)		Eksaminatorie (U20)	Eksaminatorie (U20)	

Practical Information

- Communication tools
  - Course Public Webpage (WWW) ⇔ BlackBoard (BB) (link from http://www.imada.sdu.dk/~marco/DM545/)
  - Announcements in BlackBoard
  - Personal email

- Main reading material:

B1 R. Vanderbei. Linear Programming: Foundations and Extensions. Springer US, 2008 or [B3] J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

B2 L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

A1 Robert Fourer, David M. Gay and Brian W. Kernighan, "A Modeling Language for Mathematical Programming." Management Science 36 (1990) 519-554.

  - Slides

Other references:

- many!

B12 Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

Contents

Evaluation

- 5 ECTS
- course language: English and Danish
- obligatory Assignments, pass/fail, evaluation by teacher (2 hand ins)
  practical part
  modeling + programming in AMPL
- 4 hour written exam, 7-grade scale, external censor
  theory part
  similar to exercises in class
  on June 18th

Obligatory Assignments

- Small projects (in groups of 2) must be passed to attend the written exam
- They require the use of the AMPL system + CPLEX or Gurobi
  Software available for all systems from the WWW page:
  → Software and Data → AMPL
  (Get the password)

Exercises

- Prepare them beforehand
- Best carried out in small groups (form the groups today)

Outline

What is OR?

- "**Operations Research** (or, often, Management Science, Analytics) means a **scientific approach to decision making**, which seeks to determine how best to design and operate a system, usually under conditions requiring the allocation of scarce resources."
- or short: **quantitative methods for planning and analysis**
- Components in Classical OR:
  - Real-Life Problem: e.g. Crew Rostering
  - Mathematical Model: Decision variables, constraints, objective function
  - Solution Methods (algorithms + software; general purpose, tailored)

Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
  - Fewest number of people
  - Shortest route
- Not all plans are feasible - there are constraining rules
  - Limited amount of available resources
- It can be extremely difficult to figure out what to do

Basic Idea

Build a mathematical model describing exactly what one wants, and what the "rules of the game" are. However, **what is a mathematical model and how?**

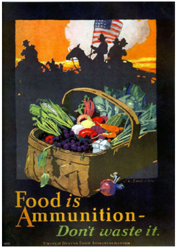
OR - The Process?

1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Outline

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear program
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
  eat enough but not too much of Vitamin A
  eat enough but not too much of Sodium
  eat enough but not too much of Calories
  ...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread, and
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50,000)

Food	Cost per serving	Vitamin A	Calories
Corn	\$0.18	107	72
2% Milk	\$0.23	500	121
Wheat Bread	\$0.05	0	65

The Mathematical Model

Parameters (given data)

F = set of foods
N = set of nutrients

a_{ij} = amount of nutrient j in food i, $\forall i \in F$, $\forall j \in N$
c_i = cost per serving of food $i, \forall i \in F$
F_{mini} = minimum number of required servings of food $i, \forall i \in F$
F_{maxi} = maximum allowable number of servings of food $i, \forall i \in F$
N_{minj} = minimum required level of nutrient $j, \forall j \in N$
N_{maxj} = maximum allowable level of nutrient $j, \forall j \in N$

Decision Variables

x_i = number of servings of food i to purchase/consume, $\forall i \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{i \in F} c_i x_i$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \forall j \in N$$

Constraint Set 2: For each nutrient $j \in N$, do not exceed the maximum allowable level.

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \forall j \in N$$

Constraint Set 3: For each food $i \in F$, select at least the minimum required number of servings

$$x_i \geq F_{mini}, \forall i \in F$$

Constraint Set 4: For each food $i \in F$, do not exceed the maximum allowable number of servings.

$$x_i \leq F_{maxi}, \forall i \in F$$

The Mathematical Model

system of equalities and inequalities

$$
\begin{aligned}
\min \quad & \sum_{i \in F} c_i x_i \\
\sum_{i \in F} a_{ij} x_i &\geq N_{minj}, \qquad \forall j \in N \\
\sum_{i \in F} a_{ij} x_i &\leq N_{maxj}, \qquad \forall j \in N \\
x_i &\geq F_{mini}, \qquad \forall i \in F \\
x_i &\leq F_{maxi}, \qquad \forall i \in F
\end{aligned}
$$

- The linear program consisted of 9 equations in 77 unknowns
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution
- The original instance:
  http://www.gams.com/modlib/libhtml/diet.htm

AMPL Model

```
# diet.mod
set NUTR;
set FOOD;
#
param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max { i in FOOD} >= f_min[i];
param n_min { NUTR } >= 0;
param n_max {j in NUTR } >= n_min[j];
param amt {NUTR,FOOD} >= 0;
#
var Buy { i in FOOD} >= f_min[i], <= f_max[i]
#
minimize total_cost: sum { i in FOOD } cost [i] * Buy[i];
subject to diet { j in NUTR }:
      n_min[j] <= sum {i in FOOD} amt[i,j] * Buy[i] <= n_max[i];
```

AMPL Model

```
# diet.dat
data;

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH MTL SPG
     TUR;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
   A 700 10000
   C 700 10000
   B1 700 10000
   B2 700 10000 ;
# %
```

```
param amt (tr):
          A C B1 B2 :=
   BEEF 60 20 10 15
   CHK 8 0 20 20
   FISH 8 10 15 10
   HAM 40 40 35 10
   MCH 15 35 15 15
   MTL 70 30 15 15
   SPG 25 50 25 15
   TUR 60 20 15 10 ;
```

History of Optimization

- Origins date back to Newton, Leibnitz, Lagrange, etc.
- In 1827 Fourier described a variable elimination method for linear inequalities, today often called Fourier-Moutzkin elimination (Motzkin, 1936). It can be turned into an LP solver
- In 1939, Kantorovich (1912-1986):
  Foundations of linear programming (Nobel prize with Koopmans on LP, 1975)
- In 1947, Dantzig (1914-2005) invented the (primal) simplex algorithm
- In 1954, Lemke: dual simplex algorithm, In 1954, Dantzig and Orchard Hays: revised simplex algorithm
- In 1970, Victor Klee and George Minty created an example that showed that the classical simplex algorithm has exponential worst-case behaviour.
- In 1979, L. Khachain found a new efficient algorithm for linear programming. It was terribly slow. (Ellipsoid method)

History of Optimization

- In 1984, Narendra Karmarkar discovered yet another new efficient algorithm for linear programming. It proved to be a strong competitor for the simplex method. (Interior point method)

- In 1951, Nonlinear Programming began with the Karush-Kuhn-Tucker Conditions
- In 1952, Commercial Applications and Software began
- In 1950s, Network Flow Theory began with the work of Ford and Fulkerson.
- In 1955, Stochastic Programming began
- In 1958, Integer Programming began by R. E. Gomory.
- In 1962, Complementary Pivot Theory

Outline

Resource Allocation

Managing a production facility

$1, 2, \ldots, n$	products
$1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
$c_j = \sigma_j - \sum_{i=1}^{n} \rho_i a_{ij}$	profit per unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Duality

Resource Valuation problem: Determine the value of the raw materials on hand such that: The company must be willing to sell the raw materials should an outside firm offer to buy them at a price consistent with the market

w_i	value of a unit of raw material i
$\sum_{i=1}^m b_i w_i$	opportunity cost (cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal is to minimize the lost opportunity cost

$$\min \sum_{i=1}^{m} b_i w_i \tag{1}$$

$$w_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} w_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(1) and (2) otherwise contradicting market

Let

$$y_i = w_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\min \sum_{i=1}^{m} y_i b_i + \sum_{i} \rho_i b_i$$

$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1 \ldots n$$

$$y_i \geq 0, \quad i = 1 \ldots m$$

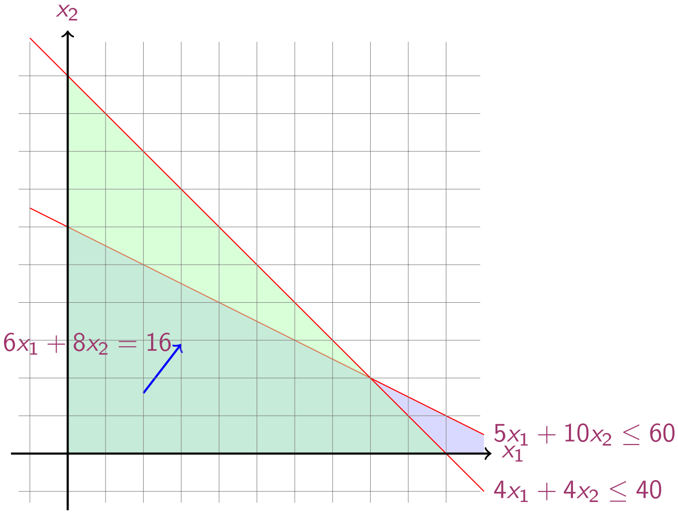
A Numerical Example

Machines A and B
products 1 and 2

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

$$
\begin{array}{lrcrcl}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & = & 60 \\
 & 4x_1 & + & 4x_2 & = & 40 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Graphical Representation



Outline

In Matrix Form

$$
\begin{array}{lccccccccccc}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
& \dots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

$$
\begin{array}{lccc}
\max & z & = & c^Tx \\
& Ax & = & b \\
& x & \geq & 0
\end{array}
$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Linear Programming

Abstract mathematical model:

Decision Variables (quantity) eg. x_1 units of 1, x_2 units of 2

Criterion (discriminate among solutions) eg. max profit: $6x_1 + 8x_2$

Constraints (limitations on resources) eg.

$5x_1 + 10x_2 = 60; 4x_1 + 4x_2 = 40; x_1 \geq 0; x_2 \geq 0$

Essential features of a Linear program:

1. continuity (later, integrality)
2. linearity $\rightsquigarrow$ proportionality + additivity
3. certainty of parameters

$$
\begin{array}{lrcll}
\text{objective func.} & \max / \min c^T \cdot x & & & c \in \mathbb{R}^n \\
\text{constraints} & A \cdot x & \gtreqless & b & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\
 & x & \geq & 0 & x \in \mathbb{R}^n, 0 \in \mathbb{R}^n
\end{array}
$$

Fourier Motzkin elimination method

Has $Ax \leq b$ a solution? (Assumption: $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^n$)
Idea:

1. transform the system into another by eliminating some variables such that the two systems have the same solutions over the remaining variables.
2. reduce to a system of constant inequalities that can be easily decided

Let x_r be the variable to eliminate
Let $M = \{1 \ldots m\}$ index the constraints
For a variable j let partition the rows of the matrix in

$$\begin{aligned} N &= \{i \in M \mid a_{ij} < 0\} \\ Z &= \{i \in M \mid a_{ij} = 0\} \\ P &= \{i \in M \mid a_{ij} > 0\} \end{aligned}$$

$$\begin{cases} x_r \geq b_{ir} - \sum_{k=1}^{r-1} a_{ik} x_k \\ x_r \leq b_{ir} - \sum_{k=1}^{r-1} a_{ik} x_k \\ \text{all other constraints}(i \in Z) \end{cases} \qquad \begin{cases} x_r \geq A_i(x_1, \ldots, x_{r-1}), & i \in N \\ x_r \leq B_i(x_1, \ldots, x_{r-1}), & i \in P \\ \text{all other constraints}(i \in Z) \end{cases}$$

Hence the original system is equivalent to

$$\begin{cases} \max\{A_i(x_1, \ldots, x_{r-1}), i \in N\} \leq x_r \leq \min\{B_i(x_1, \ldots, x_{r-1}), i \in P\} \\ \text{all other constraints}(i \in Z) \end{cases}$$

which is equivalent to

$$\begin{cases} A_i(x_1, \ldots, x_{r-1}) \leq B_j(x_1, \ldots, x_{r-1}) \qquad i \in N, j \in P \\ \text{all other constraints}(i \in Z) \end{cases}$$

we eliminated x_r but:

$$\begin{cases} |N| \cdot |P| \text{ inequalities} \\ |Z| \text{ inequalities} \end{cases}$$

after d iterations if $|P| = |N| = n$ exponential growth: $(n/2)^{2^d}$

Example

$$
\begin{array}{rcrcr}
-7x_1 & + & 6x_2 & \leq & 25 \\
x_1 & - & 5x_2 & \leq & 1 \\
x_1 & & & \leq & 7 \\
-x_1 & + & 2x_2 & \leq & 12 \\
-x_1 & - & 3x_2 & \leq & 1 \\
2x_1 & - & x_2 & \leq & 10
\end{array}
$$

x_2 variable to eliminate
$N = \{2, 5, 6\}, Z = \{3\}, P = \{1, 4\}$
$|Z \cup (N \times P)| = 7$ constraints

Notation

- $\mathbb{N}$ natural numbers, $\mathbb{Z}$ integer numbers, $\mathbb{Q}$ rational numbers, $\mathbb{R}$ real numbers

- column vector and matrices
  scalar product: $y^T x = \sum_{i=1}^{n} y_i x_i$

- linear combination

$$
\begin{aligned}
x &\in \mathbb{R}^k \\
x_1, \ldots, x_k &\in \mathbb{R} \qquad x = \sum_{i=1}^{k} \lambda_i x_i \\
\lambda = (\lambda_1, \ldots, \lambda_k)^T &\in \mathbb{R}^k
\end{aligned}
$$

  moreover:

$\lambda \geq 0$	conic combination
$\lambda^T 1 = 1 \quad (\sum_{i=1}^{k} \lambda_i = 1)$	affine combination
$\lambda \geq 0$ and $\lambda^T 1 = 1$	convex combination

Notation

- set S is linear independent if no element of it can be expressed as combination of the others
  Eg: $S \subseteq \mathbb{R} \implies$ max n lin. indep.

- rank of a matrix for columns (= for rows)
  if (m, n)-matrix has rank $= \min\{m, n\}$ then the matrix is full rank
  if (n, n)-matrix is full rank is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

$$G = \{x \in \mathbb{R}^n \mid a^T x = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

$$H = \{x \in \mathbb{R}^n \mid a^T x \leq \alpha\}$$

  ($a^T x = \alpha$ is a supporting hyperplane of H)

Notation

- a set $S \subset \mathbb{R}$ is a polyhedron if $\exists m \in \mathbb{Z}^{+}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m}$:

$$P = \{x \in \mathbb{R} \mid Ax \leq b\} = \cap_{i=1}^{m} \{x \in \mathbb{R}^{n} \mid A_{i}.x \leq b_{i}\}$$

- a polyhedron P is a polytope if it is bounded: $\exists B \in \mathbb{R}, B > 0$:

$$p \subseteq \{x \in \mathbb{R}^{n} \mid \| x \| \leq B\}$$

- every polyhedron $P \neq \mathbb{R}^{n}$ is determined by finitely many halfspaces

Summary

1. Course Introduction
2. Introduction
3. Diet Problem
4. Resource Allocation
5. Definitions and Basics