DM545

Linear and Integer Programming

Lecture 2
The Simplex Method

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Outline

Outline

Linear Programming

Abstract mathematical model:

Decision Variables

Criterion

Constraints

$$
\begin{array}{rrcll}
\text{objective func.} & \max / \min c^T \cdot x & & & c \in \mathbb{R}^n \\
\text{constraints} & A \cdot x & \begin{smallmatrix} \geq \\ = \\ \leq \end{smallmatrix} & b & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\
 & x & \geq & 0 & x \in \mathbb{R}^n, 0 \in \mathbb{R}^n
\end{array}
$$

- Any vector $x \in \mathbb{R}^n$ satisfying all constraints is a feasible solution.
- Each $x^* \in \mathbb{R}^n$ that gives the best possible value for $c^T x$ among all feasible x is an optimal solution or optimum
- The value $c^T x^*$ is the optimum value

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & c^Tx \\
 & Ax & = & b \\
 & x & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Definitions

- $\mathbb{N}$ natural numbers, $\mathbb{Z}$ integer numbers, $\mathbb{Q}$ rational numbers, $\mathbb{R}$ real numbers

- column vector and matrices
  scalar product: $y^T x = \sum_{i=1}^n y_i x_i$

- linear combination

$$x \in \mathbb{R}^n \qquad x_1, \ldots, x_k \in \mathbb{R} \qquad \lambda = (\lambda_1, \ldots, \lambda_k)^T \in \mathbb{R}^k \qquad x = \sum_{i=1}^k \lambda_i x_i$$

  moreover:

$\lambda \geq 0$	conic combination
$\lambda^T 1 = 1 \quad (\sum_{i=1}^k \lambda_i = 1)$	affine combination
$\lambda \geq 0$ and $\lambda^T 1 = 1$	convex combination

Definitions

- set S is linear independent if no element of it can be expressed as combination of the others
  Eg: $S \subseteq \mathbb{R} \implies$ max n lin. indep.

- rank of a matrix for columns (= for rows)
  if (m, n)-matrix has rank $= \min\{m, n\}$ then the matrix is full rank
  if (n, n)-matrix is full rank is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

$$G = \{x \in \mathbb{R}^n \mid a^T x = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

$$H = \{x \in \mathbb{R}^n \mid a^T x \leq \alpha\}$$

  ($a^T x = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subseteq\in \mathbb{R}$ is a polyhedron if $\exists m \in \mathbb{Z}^+, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$:

$$P = \{x \in \mathbb{R} \mid Ax \leq b\} = \cap_{i=1}^m \{x \in \mathbb{R}^n \mid A_i.x \leq b_i\}$$

- a polyhedron P is a polytope if it is bounded: $\exists B \in \mathbb{R}, B > 0$:

$$p \subseteq \{x \in \mathbb{R}^n \mid \| x \| \leq B\}$$

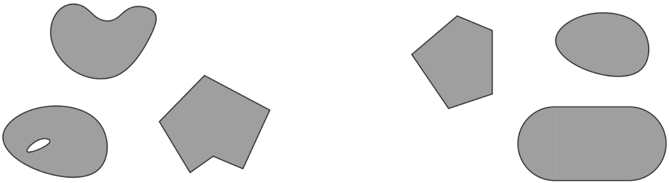
- Theorem: every polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

Definitions

- General optimization problem:
  $\max\{\varphi(x) \mid x \in F\}$, F is feasible region for x
- If A and b are rational numbers, $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a rational polyhedron
- convex set: if $x, y \in P$ and $0 \leq \lambda \leq 1$ then $\lambda x + (1 - \lambda)y \in P$
- convex function if its epigraph $\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is a convex set or $f : X \to \mathbb{R}$, if $\forall x, y \in X, \lambda \in [0, 1]$ it holds that
  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$

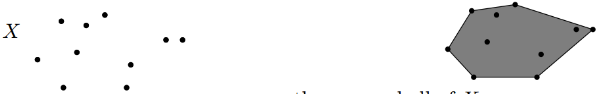
Definitions

[<+->]



nonconvex convex

- Given a set of points $X \subseteq \mathbb{R}^n$ the convex hull $\mathbf{conv}(X)$ is the convex linear combination of the points



the convex hull of X

Definitions

- A face of P is either P itself or the intersection of P with a supporting hyperplane
- A point x for which $\{x\}$ is a face is called a vertex of P and also a basic solution of $Ax \leq b$
- A facet is a maximal face distinct from P
  $cx \leq d$ is facet defining if $cx = d$ is a supporting hyperplane of P

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Task:

1. decide that $\{x \in \mathbb{R}^n; Ax \leq b\}$ is empty (prob. infeasible), or
2. find a column vector $x \in \mathbb{R}^n$ such that $Ax \leq b$ and $c^T x$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $x \in \mathbb{R}^n$ with $Ax \leq b$ and $c^T x > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and $\exists$ solution
    1. one solution
    2. infinite solution
3. $F \neq \emptyset$ and $\nexists$ solution

Linear Programming and Linear Algebra

- Linear algebra: linear equations (Gaussian elimination)
- Integer linear algebra: linear diophantine equations
- Linear programming: linear inequalities (simplex method)
- Integer linear programming: linear diophantine inequalities

Outline

Fundamental Theorem of LP

Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{c^T x \mid x \in P\} \ \textit{where } P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

If P is a bounded polyhedron and not empty (ie, a polytope) and x^* is an optimal solution to the problem, then:

- *x^* is an extreme point (vertex) of P, or*
- *x^* lies on a face $F \subset P$ of optimal solution*

Proof:

- assume x^* not a vertex of P then $\exists$ a ball around it still in P. Show that a point in the ball has better cost
- if x^* is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

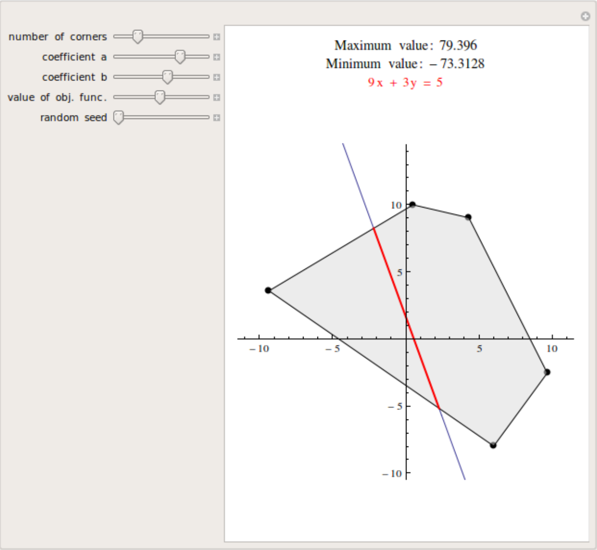
Implications:

- the optimal solution is at the intersection of hyperplanes supporting halfspaces.
- hence finitely many possibilities
- Solution method: write all inequalities as equalities and solve all $\binom{n}{m}$ systems of linear equalities
- for each point we need then to check if feasible and if best in cost.
- each system is solved by Gaussian elimination

Simplex Method

1. find a solution that is at the intersection of some n hyperplanes
2. try systematically to produce the other points by exchanging one hyperplane with another
3. check optimality, proof provided by duality theory

Demo


number of corners
coefficient a
coefficient b
value of obj. func.
random seed
Maximum value: 79.396
Minimum value: – 73.3128
9 x + 3 y = 5
10
5
– 5
– 10
– 10
– 5
5
10

Outline

Gaussian Elimination

1. Forward elimination
   reduces the system to triangular (row echelon) form (or degenerate)
   elementary row operations (or LU decomposition)
2. back substitution

Example:

$$\begin{array}{rcrcrcrr} 2x & + & y & - & z & = & 8 & (I) \\ -3x & - & y & + & 2z & = & -11 & (II) \\ -2x & + & y & + & 2z & = & -3 & (III) \end{array}$$

```
|-----------+---+-----+-----+---|
|           | 2 |   1 |  -1 | 8 |
| 3/2 I+II  | 0 | 1/2 | 1/2 | 1 |
| I+III     | 0 |   2 |   1 | 5 |
|-----------+---+-----+-----+---|

|-----------+---+-----+-----+---|
|           | 2 |   1 |  -1 | 8 |
|           | 0 | 1/2 | 1/2 | 1 |
| -4 II+III | 0 |   0 |  -1 | 1 |
|-----------+---+-----+-----+---|

|---+-----+-----+---|
| 2 |   1 |  -1 | 8 |
| 0 | 1/2 | 1/2 | 1 |
| 0 |   0 |  -1 | 1 |
|---+-----+-----+---|

|---+---+---+----|
| 1 | 0 | 0 |  2 | => x=2
| 0 | 1 | 0 |  3 | => y=3
| 0 | 0 | 1 | -1 | => z=-1
|---+---+---+----|
```

Polynomial time $O(n^2m)$ but needs to guarantee that all the numbers during the run can be represented by polynomially bounded bits

Outline

1. Definitions and Basics

2. Fundamental Theorem of LP

3. Gaussian Elimination

4. Simplex Method
   - Standard Form
   - Basic Feasible Solutions
   - Algorithm
   - Tableaux and Dictionaries

A Numerical Example

$$\begin{array}{lrll} \max & \sum_{j=1}^{n} c_j x_j & & \\ & \sum_{j=1}^{n} a_{ij} x_j & \leq b_i, & i = 1, \ldots, m \\ & x_j & \geq 0, & j = 1, \ldots, n \end{array}$$

$$\begin{array}{lrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \max & c^T x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

Standard Form

Each linear program can be converted in the form:

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leq b \\ & x \in \mathbb{R}^n \end{aligned}$$

$c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

- if equations, then put two constraints, $ax \leq b$ and $ax \geq b$
- if $ax \geq b$ then $-ax \leq -b$
- if $\min c^T x$ then $\max(-c^T x)$

and then be put in standard (or equational) form

$$\begin{aligned} \max \quad & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

1. "$=$" constraints
2. $x \geq 0$ nonnegativity constraints
3. ($b \geq 0$)
4. max

Transformation into Std Form

Every LP can be transformed in std. form

1. introduce slack variables (or surplus)

$$\begin{array}{ccccccc} 5x_1 & + & 10x_2 & + & x_3 & = & 60 \\ 4x_1 & + & 4x_2 & + & x_4 & = & 40 \end{array}$$

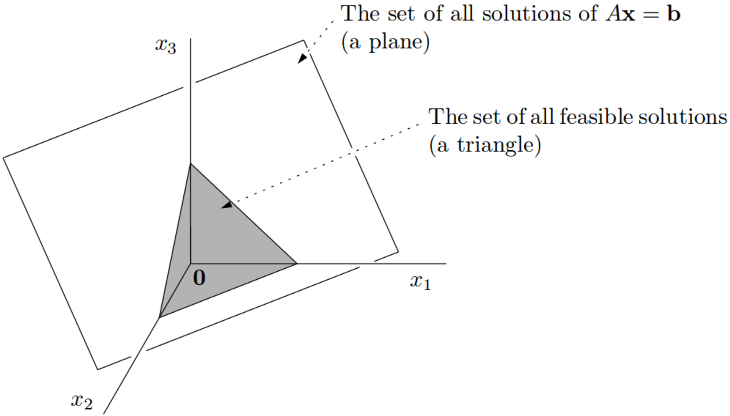
2. if $x_1 \gtreqless 0$ then $\begin{array}{l} x_1 = x_1' - x_1'' \\ x_1' \geq 0 \\ x_1'' \geq 0 \end{array}$

3. $(b \geq 0)$

4. $\min c^T x \equiv \max(-c^T x)$

LP in $m \times m$ converted into LP with at most $(m + 2n)$ variables and m equations

Geometry


The set of all solutions of $A\mathbf{x} = \mathbf{b}$
(a plane)
The set of all feasible solutions
(a triangle)
x_3
$\mathbf{0}$
x_1
x_2

- $Ax = b$ is a system of equations that we can solve by Gaussian elimination

- Elementary row operations of $\begin{bmatrix} A & | & b \end{bmatrix}$ do not affect set of feasible solutions
  - multiplying all entries in some row of $\begin{bmatrix} A & | & b \end{bmatrix}$ by a nonzero real number λ
  - replacing the ith row of $\begin{bmatrix} A & | & b \end{bmatrix}$ by the sum of the ith row and jth row for some $i \neq j$

- We assume $\mathrm{rank}(\begin{bmatrix} A & | & b \end{bmatrix}) = \mathrm{rank}(A) = m$, ie, rows of A are linearly independent
  otherwise, remove linear dependent rows

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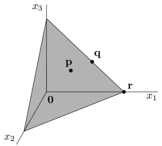
Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:
Let $B = \{1 \dots m\}$, $N = \{m+1 \dots n+m\}$ be subsets of columns
A_B is columns of A indexed by B:

Definition

$x \in \mathbb{R}^n$ is a basic feasible solution of the linear program $\max\{c^T x \mid Ax = b, x \geq 0\}$ for an index set B if:

- $x_j = 0 \, \forall j \notin B$
- the square matrix A_B is nonsingular, ie, all columns indexed by B are lin. indep.
- $x_B = A_B^{-1} b$ is nonnegative, ie, $x_B \geq 0$ (feasibility)

We call $x_j, j \in B$ basic variables and remaining variables nonbasic variables.

Theorem

A basic feasible solution is uniquely determined by the set B.

Proof:

$$
\begin{aligned}
Ax = & A_B x_B + A_N x_N = b \\
& x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\
& x_B = A_B^{-1} b \qquad\qquad A_B \text{ is singular hence one solution}
\end{aligned}
$$

Note: we call B a (feasible) basis

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

Theorem

Let P be a (convex) polyhedron from LP in std. form. For a point $v \in P$ the following are equivalent:

(i) *v is an extreme point (vertex) of P*

(ii) *v is a basic feasible solution of LP*

Proof: by recognizing that vertices of P are linear independent and such are the columns in A_B

Theorem

Let $LP = \max\{c^T x \mid Ax = b, x \geq 0\}$ be feasible and bounded, then the optimal solution is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming

Idea for solution method:
examine all basic solutions.
There are finitely many: $\binom{m+n}{m}$.
However, if $n = m$ then $\binom{2m}{m} \approx 4^m$.

Outline

Simplex Method

$$\max \quad z = \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical std. form: one decision variable is isolated in each constraint and does not appear in the other constraints or in the obj. func.

It gives immediately a feasible solution:

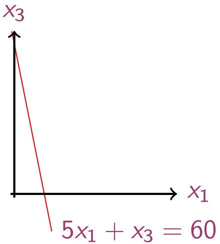
$$x_1 = 0,\ x_2 = 0,\ x_3 = 60,\ x_4 = 40$$

Is it optimal? Look at signs in $z \rightsquigarrow$ if positive then an increase would improve.

Let's try to increase a promising variable, ie, x_1, one with positive coefficient in z (is the best choice?)

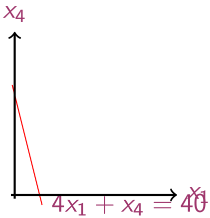
$$5x_1 + x_3 = 60$$
$$x_1 = \frac{60}{5} - \frac{x_3}{5}$$
$$x_3 = 60 - 5x_1 \geq 0$$

If $x_1 > 12$ then $x_3 < 0$



$$4x_1 + x_4 = 40$$
$$x_1 = \frac{40}{4} - \frac{x_4}{4}$$
$$x_4 = 40 - 4x_1 \geq 0$$

If $x_1 > 10$ then $x_4 < 0$



we can take the minimum of the two $\rightsquigarrow$ x_1 increased to 10
x_4 exits the basis and x_1 enters

Simplex Tableau

First simplex tableau:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

we want to reach this new tableau

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	?	1	?	0	?
x_1	1	?	0	?	0	?
	0	?	0	?	1	?

Pivot operation:

1. Choose pivot:

   column: one with positive coefficient in obj. func. (to discuss later)

   row: ratio between coefficient b and pivot column: choose the one with smallest ratio:

$$\theta = \min_i \left\{ \frac{b_i}{a_{is}} : a_{is} > 0 \right\}, \qquad \theta \text{ increase value of entering var.}$$

2. elementary row operations to update the tableau

- x_4 leaves the basis, x_1 enters the basis
  - Divide row pivot by pivot
  - Send to zero the coefficient in the pivot column of the first row
  - Send to zero the coefficient of the pivot column in the third (cost) row

	x1	x2	x3	x4	-z	b
I'=I-5II'	0	5	1	-5/4	0	10
II'=II/4	1	1	0	1/4	0	10
III'=III-6II'	0	2	0	-6/4	1	-60

From the last row we read: $2x_2 - 3/2x_4 - z = -60$, that is:
$z = 60 + 2x_2 - 3/2x_4$.
Since x_2 and x_4 are nonbasic we have $z = 60$ and
$x_1 = 10, x_2 = 0, x_3 = 10, x_4 = 0$.

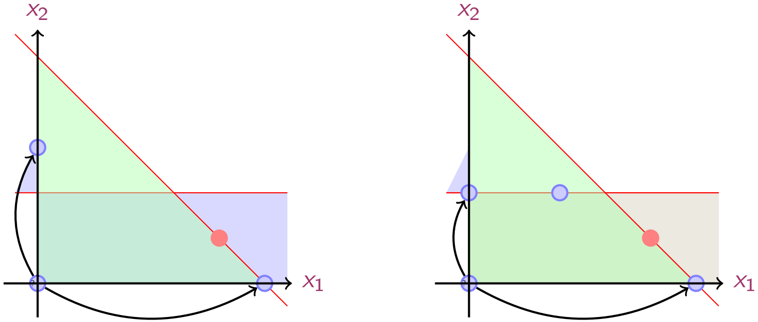
- Done? No! Let x_2 enter the basis

	x1	x2	x3	x4	-z	b
I'=I/5	0	1	1/5	-1/4	0	2
II'=II-I'	1	0	-1/5	1/2	0	8
III'=III-2I'	0	0	-2/5	-1	1	-64

Optimality:
The basic solution is optimal when the coefficient of the nonbasic variables (reduced costs) in the corresponding simplex tableau are nonpositive, ie, such that:

$$\bar{c}_N \leq 0$$

Graphical Representation



Outline

Tableaux and Dictionaries

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, m$$

$$z = \sum_{j=1}^{n} c_j x_j$$

Tableau

$$\left[\begin{array}{c:c:c:c} I & \bar{A}_N & 0 & \bar{b} \\ \hdashline 0 & \bar{c}_N & 1 & -\bar{d} \end{array}\right]$$

$\bar{c}_N$ reduced costs

Dictionary

$$x_r = \bar{b}_r - \sum_{s \notin B} \bar{a}_{rs} x_s, \quad r \in B$$

$$z = \bar{d} + \sum_{s \notin B} \bar{c}_s x_s$$

pivot op.:
choose col with r.c. >0
choose row with negative sign
update: express entering variable
and substitute in other rows

Example

$$
\begin{array}{rrcrcr}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

$$
\begin{array}{rcrcrcr}
x_3 & = & 60 & - & 5x_1 & - & 10x_2 \\
x_4 & = & 40 & - & 4x_1 & - & 4x_2 \\
\hline
z & = & & + & 6x_1 & + & 8x_2
\end{array}
$$

...

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	2
x_1	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

$$
\begin{array}{rcrcrcr}
x_1 & = & 2 & - & 1/5x_3 & + & 1/4x_4 \\
x_2 & = & 8 & + & 1/5x_3 & - & 1/2x_4 \\
\hline
z & = & 64 & - & 2/5x_3 & - & 1x_4
\end{array}
$$

Summary

1. Definitions and Basics
2. Fundamental Theorem of LP
3. Gaussian Elimination
4. Simplex Method
   - Standard Form
   - Basic Feasible Solutions
   - Algorithm
   - Tableaux and Dictionaries

Exception Handling

1. Unboundedness
2. More than one solution
3. Degeneracies
   - benign
   - cycling
4. Infeasible starting