DM545

Linear and Integer Programming

Lecture 3
The Simplex Method: Exception Handling

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Outline

Outline

Exception Handling

1. Unboundedness
2. More than one solution
3. Degeneracies
   - benign
   - cycling
4. Infeasible starting

Unboundedness

$$\begin{array}{rrcrcl} \max & 2x_1 & + & x_2 & & \\ & & & x_2 & \leq & 5 \\ & -x_1 & + & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

- Initial tableau

```
|    | x1 | x2 | x3 | x4 | -z | b |
|----+----+----+----+----+----+---|
| x3 |  0 |  1 |  1 |  0 |  0 | 5 |
| x4 | -1 |  1 |  0 |  1 |  0 | 1 |
|----+----+----+----+----+----+---|
|    |  2 |  1 |  0 |  0 |  1 | 0 |
```

- x_2 entering, x_4 leaving

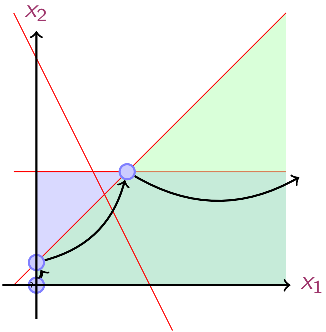
```
|              | x1 | x2 | x3 | x4 | -z |  b |
|--------------+----+----+----+----+----+----|
| II'=II-I'    |  1 |  0 |  1 | -1 |  0 |  4 |
| I'=I         | -1 |  1 |  0 |  1 |  0 |  1 |
|--------------+----+----+----+----+----+----|
| III'=III-I'  |  3 |  0 |  0 | -1 |  1 | -1 |
```

$-x_1 + x_2 + x_4 = 1$, x_1 can increase without restriction,
$\theta = \min\{\frac{b_i}{a_{is}} : a_{is} > 0, i = 1 \ldots, n\}$

- x_1 entering, x_3 leaving

```
|               | x1 | x2 | x3 | x4 | -z |   b |
|---------------+----+----+----+----+----+-----|
| I'=I          |  1 |  0 |  1 | -1 |  0 |   4 |
| II'=II+I'     |  0 |  1 |  1 |  0 |  0 |   5 |
|---------------+----+----+----+----+----+-----|
| III'=III-3I'  |  0 |  0 | -3 |  2 |  1 | -13 |
```

  x_4 was already in basis but for both I and II ($x_3 + 0x_4 = 5$), x_4 can increase arbitrarily



∞ solutions

$$
\begin{array}{lrcrcl}
\max & x_1 & + & x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

- Initial tableau

```
|    | x1 | x2 | x3 | x4 | -z |  b |
|----+----+----+----+----+----+----|
| x3 |  5 | 10 |  1 |  0 |  0 | 60 |
| x4 |  4 |  4 |  0 |  1 |  0 | 40 |
|----+----+----+----+----+----+----|
|    |  1 |  1 |  0 |  0 |  1 |  0 |
```

- x_2 enters, x_3 leaves

```
|             |  x1 | x2 | x3   | x4 | -z |  b |
|-------------+-----+----+------+----+----+----|
| I'=I/10     | 1/2 |  1 | 1/10 |  0 |  0 |  6 |
| II'=II-4Ix4 |   2 |  0 | -2/5 |  1 |  0 | 16 |
|-------------+-----+----+------+----+----+----|
| III'=III-I  | 1/2 |  0 | -1/6 |  0 |  1 | -6 |
```

- x_1 enters, x_4 leaves

	x1	x2	x3	x4	-z	b
I'=I-II'/2	0	1	1/5	-1/4	0	2
II'=II/2	1	0	-1/5	1/2	0	8
III'=III-II'/2	0	0	0	-1/4	1	-10

  $\vec{x} = (8, 2, 0, 0), z = 10$
  nonbasic variables typically have reduced costs $\neq 0$. Here x_3 has r.c. $= 0$. Let's make it enter the basis

- x_3 enters, x_2 leaves

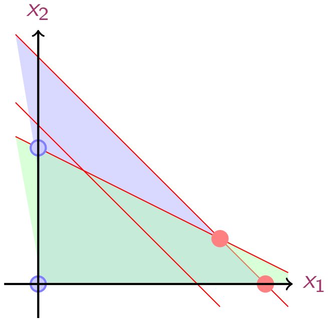
	x1	x2	x3	x4	-z	b
I'=5I	0	5	1	-5/4	0	10
II'=II+I'/5	1	1	0	4	0	10
III'=III	0	0	0	-/14	1	-10

  $\vec{x} = (10, 0, 10, 0), z = 10$
  There are 2 optimal solutions $\rightsquigarrow$ all their convex combinations are optimal solutions:

$$\vec{x} = \sum_i \alpha_i \vec{x}_i$$
$$\alpha_i \geq 0$$
$$\sum_i \alpha_i = 1$$

$$\vec{x}^1 = (8, 2, 0, 0)$$
$$\vec{x}^2 = (10, 0, 10, 0)$$

$$x_1 = 8\alpha + 10(1 - \alpha)$$
$$x_2 = 2\alpha$$
$$x_3 = 10(1 - \alpha)$$
$$x_4 = 0$$



Degeneracy

$$
\begin{array}{lrcrcr}
\max & & & x_2 & & \\
& -x_1 & + & x_2 & \leq & 0 \\
& x_1 & & & \leq & 2 \\
& & & x_1, x_2 & \geq & 0
\end{array}
$$

- Initial tableau

```
|    | x1 | x2 | x3 | x4 | -z | b |
|----+----+----+----+----+----+---|
| x3 | -1 |  1 |  1 |  0 |  0 | 0 |
| x4 |  1 |  0 |  0 |  1 |  0 | 2 |
|----+----+----+----+----+----+---|
|    |  0 |  1 |  0 |  0 |  1 | 0 |
```

$b_i = 0$ (one basic var. is zero) might lead to cycling

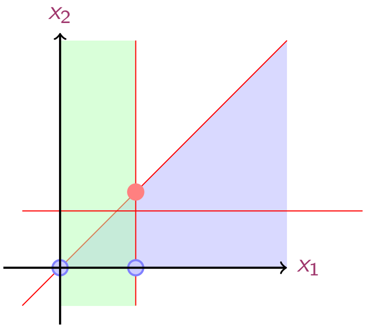
- degenerate pivot step: not improving, the entering variable stays at zero

```
|   | x1 | x2 | x3 | x4 | -z | b |
|---+----+----+----+----+----+---|
|   | -1 |  1 |  1 |  0 |  0 | 0 |
|   |  1 |  0 |  0 |  1 |  0 | 2 |
|---+----+----+----+----+----+---|
|   |  1 |  0 | -1 |  0 |  1 | 0 |
```

- now nondegenerate:

```
|   | x1 | x2 | x3 | x4 | -z |  b |
|---+----+----+----+----+----+----|
|   |  0 |  1 |  0 |  1 |  0 |  2 |
|   |  1 |  0 |  0 |  1 |  0 |  2 |
|---+----+----+----+----+----+----|
|   |  0 |  0 | -1 | -1 |  1 | -2 |
```

$x_1 = 2, x_2 = 2, z = 2$



$\geq n+1$ constraints meet at a vertex

Def: Improving variable, one with positive reduced cost

Under certain pivoting rules cycling can happen. So far we chose an arbitrary improving variable to enter.

Degenerate conditions may appear often in practice but cycling is rare and some pivoting rules prevent cycling. (Ex. 11 Sheet 1 shows the smallest possible example)

Theorem

If the simplex fails to terminate, then it must cycle.

Proof:

- there is a finite number of basis and simplex chooses to always increase the cost
- hence the only situation for not terminating is that a basis must appear again. Two dictionaries with the same basis are the same (related to uniqueness of basic solutions)

Pivot Rules

Rules for breaking ties in selecting entering improving variables (more important than selecting leaving variables)

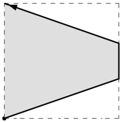
- Largest Coefficient: the improving var with largest coefficient in last row of the tableau.
  Original Dantzig's rule, can cycle
- Largest increase: absolute improvement: $\text{argmax}_j \{c_j \theta_j\}$
  computationally more costly
- Steepest edge the improving var whose entering into the basis moves the current basic feasible sol in a direction closest to the direction of the vector $\vec{c}$ (ie, minimizes the cosine of the angle between the two vectors):

$$\frac{c^T(x_{\text{new}} - x_{\text{old}})}{\| x_{\text{new}} - x_{\text{old}} \|}$$

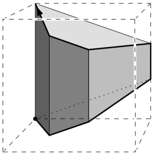
- Bland's rule choose the improving var with the lowest index and if there are more than one, the leaving variable with the lowest index
  Prevents cycling but is slow
- Random edge select var uniformly at random among the improving ones
- Perturbation method perturb values of b_i terms to avoid $b_i = 0$, which must occur for cycling.
  To avoid cancellations: $0 < \epsilon_m \ll \epsilon_{m-1} \ll \cdots \ll \epsilon_1 \ll 1$
  can be shown to be the same as lexicographic method, which prevents cycling

Efficiency of Simplex Method

- In practice between $2m$ and $3m$ iterations
- Klee and Minty 1978 constructed an example that requires $2^n - 1$ iterations:



$n = 2$



$n = 3$

- random shuffle of indexes + lowest index for entering + lexicographic for leaving: expected iterations $< e^{C\sqrt{n \ln n}}$
- Clairvoyant's rule: shortest possible sequence of steps
  Hirsh conjecture $O(n)$ but best known $n^{1+\ln n}$
- unknown if there exists a pivot rule that leads to polynomial time.
- smoothed complexity results

Initial Infeasibility

$$\begin{array}{rrcrcl} \max & x_1 & - & x_2 & & \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rrcrcrcrcl} \max & x_1 & - & x_2 & & & & & & \\ & x_1 & + & x_2 & + & x_3 & & & = & 2 \\ & 2x_1 & + & 2x_2 & & & - & x_4 & = & 5 \\ & & & & & & x_1, x_2, x_3, & x_4 & \geq & 0 \end{array}$$

- Initial tableau

```
|    | x1 | x2 | x3 | x4 | -z | b |
|----+----+----+----+----+----+---|
| x3 |  1 |  1 |  1 |  0 |  0 | 2 |
| x4 |  2 |  2 |  0 | -1 |  0 | 5 |
|----+----+----+----+----+----+---|
|    |  1 | -1 |  0 |  0 |  1 | 0 |
```

$\rightsquigarrow$ we do not have an initial basic feasible solution!!

In general finding any feasible solution is difficult as finding an optimal solution, otherwise we could do binary search

Auxiliary Problem (I Phase of Simplex)
We introduce auxiliary variables:

$$
\begin{array}{lcccccccccc}
\max & -x_5 & \equiv & \min x_5 \\
& x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\
& 2x_1 & + & 2x_2 & & & - & x_4 & + & x_5 & = & 5 \\
& & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

if $w^* = 0$ then $x_5 = 0$ and the two problems are equivalent
if $w^* > 0$ then not possible to set x_5 to zero.

- Initial tableau

```
|    | x1 | x2 | x3 | x4 | x5 | -z | -w | b |
|----+----+----+----+----+----+----+----+---|
|    |  1 |  1 |  1 |  0 |  0 |  0 |  0 | 2 |
|    |  2 |  2 |  0 | -1 |  1 |  0 |  0 | 5 |
| z  |  1 | -1 |  0 |  0 |  0 |  1 |  0 | 0 |
|----+----+----+----+----+----+----+----+---|
| w  |  0 |  0 |  0 |  0 | -1 |  0 |  1 | 0 |
```

Keep z always in basis

- we reach a canonical form simply by letting x_5 enter the basis:

```
|       | x1 | x2 | x3 | x4 | x5 | -z | -w | b |
|-------+----+----+----+----+----+----+----+---|
|       |  1 |  1 |  1 |  0 |  0 |  0 |  0 | 2 |
|       |  2 |  2 |  0 | -1 |  1 |  0 |  0 | 5 |
| z     |  1 | -1 |  0 |  0 |  0 |  1 |  0 | 0 |
|-------+----+----+----+----+----+----+----+---|
| IV+II |  2 |  2 |  0 | -1 |  0 |  0 |  1 | 5 |
```

  now we have a basic feasible solution!

- x_1 enters, x_3 leaves

```
|        | x1 | x2 | x3 | x4 | x5 | -z | -w |  b |
|--------+----+----+----+----+----+----+----+----|
|        |  1 |  1 |  1 |  0 |  0 |  0 |  0 |  2 |
| II-2I' |  0 |  0 | -2 | -1 |  1 |  0 |  0 |  1 |
| III-I' |  0 | -2 | -1 |  0 |  0 |  1 |  0 | -2 |
|--------+----+----+----+----+----+----+----+----|
| IV-2I' |  0 |  0 | -2 | -1 |  0 |  0 |  1 |  1 |
```

  $w^* = -1$ then no solution with $X_5 = 0$ exists then no feasible solution to initial problem

Initial Infeasibility - Another Example

$$
\begin{array}{lrcrcl}
\max & x_1 & - & x_2 & & \\
 & x_1 & + & x_2 & \leq & 2 \\
 & 2x_1 & + & 2x_2 & \geq & 2 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{lrcrcrcrcl}
\max & x_1 & - & x_2 & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & = & 2 \\
 & 2x_1 & + & 2x_2 & & & - & x_4 & = & 2 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

Auxiliary problem (I phase):

$$
\begin{array}{lrcrcrcrcrcl}
w = \max & -x_5 & \equiv & \min x_5 & & & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\
 & 2x_1 & + & 2x_2 & & & - & x_4 & + & x_5 & = & 2 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

- Initial tableau

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
z	1	-1	0	0	0	1	0	0
w	0	0	0	0	-1	0	1	0

$\leadsto$ we do not have an initial basic feasible solution.

- set in canonical form:

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
z	1	-1	0	0	0	1	0	0
IV+II	2	2	0	-1	0	0	1	2

- x_1 enters, x_5 leaves

	x1	x2	x3	x4	x5	-z	-w	b
	0	0	1	1/2	-1/2	0	0	1
	1	1	0	-1/2	1/2	0	0	1
z	0	-2	0	1/2	-1/2	1	0	-1
w	0	0	0	0	-1	0	1	0

$w^* = 0$ hence $x_5 = 0$ we have a starting feasible solution for the initial problem.

- (II phase) We keep only what we need:

```
|    | x1 | x2 | x3 | x4   | -z |  b |
|----+----+----+----+------+----+----|
|    |  0 |  0 |  1 | 1/2  |  0 |  1 |
|    |  1 |  1 |  0 | -1/2 |  0 |  1 |
|----+----+----+----+------+----+----|
| z  |  0 | -2 |  0 | 1/2  |  1 | -1 |
```

-
```
|    | x1 | x2 | x3 | x4 | -z |  b |
|----+----+----+----+----+----+----|
|    |  0 |  0 |  2 |  1 |  0 |  2 |
|    |  1 |  1 |  1 |  0 |  0 |  2 |
|----+----+----+----+----+----+----|
| z  |  0 | -2 | -1 |  0 |  1 | -2 |
```

Optimal solution: $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 2$, $z = 2$.

In Dictionary Form

$$\begin{array}{rcrcl} \max & x_1 & - & x_2 & \\ & x_1 & + & x_2 & \leq 2 \\ & 2x_1 & + & 2x_2 & \geq 5 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

$$\begin{array}{rcrcrcr} x_3 & = & 2 & - & x_1 & - & x_2 \\ x_4 & = & -5 & + & 2x_1 & + & 2x_2 \\ \hline z & = & x_1 & + & x_2 & & \end{array}$$

sol. infeasible

We introduce corrections of infeasibility

$$\begin{array}{rcrcrcrcl} \max & -x_0 & \equiv & \min & x_0 & & & & \\ & x_1 & + & x_2 & - & x_0 & \leq & 2 \\ & 2x_1 & + & 2x_2 & - & x_0 & \geq & 5 \\ & & & x_1, x_2, x_0 & \geq & 0 & & \end{array}$$

$$\begin{array}{rcrcrcrcr} x_3 & = & 2 & - & x_1 & - & x_2 & + & x_0 \\ x_4 & = & -5 & + & 2x_1 & + & 2x_2 & + & x_0 \\ \hline z & = & & & & & & - & x_0 \end{array}$$

It is still infeasible but it can be made feasible by letting x_0 enter the basis which variable should leave?
the most infeasible: the var with the b term whose negative value has the largest magnitude

Summary

Simplex Method:
Phase I + Phase II

1. Pivot operation
2. Optimality test

(Handling exceptions)