DM545

Linear and Integer Programming

Lecture 4
Duality Theory

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Outline

1. Derivation and Motivation

2. Theory

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1. Derivation and Motivation

2. Theory

Dual Problem

A dual variable y_i associated to each constraint:

Primal problem:

$$\begin{aligned} \max \quad z &= c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= b^T y \\ Ay &\geq c \\ y &\geq 0 \end{aligned}$$

Bounding approach

$$\begin{array}{lrcrcrcl} \max & 4x_1 & + & x_2 & + & 3x_3 & & \\ & x_1 & + & 4x_2 & & & \leq & 1 \\ & 3x_1 & + & x_2 & + & x_3 & \leq & 3 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

a feasible solution is a lower bound but how good?
By tentatives:

$$(x_1, x_2, x_3) = (1, 0, 0) \rightsquigarrow z^* \geq 4$$
$$(x_1, x_2, x_3) = (0, 0, 3) \rightsquigarrow z^* \geq 9$$

What about upper bounds?

$$\begin{array}{rrcrcrcl} 2 \cdot (& x_1 & + & 4x_2 & &) & \leq & 2(1) \\ +3 \cdot (& 3x_1 & + & x_2 & + & x_3) & \leq & 3(3) \\ \hline & & & \multicolumn{3}{r}{11x_1 + 5x_2 + 3x_3} & \leq & 11 \end{array}$$

$$\begin{array}{ccccc} 4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 5x_2 + 3x_3 & \leq & 11 \\ c^T x & \leq & y^T Ax & \leq & y^T b \end{array}$$

Hence $z^* \leq 11$. Is this the best upper bound we can find?

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$\begin{array}{rcccccl} y_1 \cdot (& x_1 & + & 4x_2 & &) & \leq & y_1(1) \\ +y_2 \cdot (& 3x_1 & + & x_2 & + & x_3) & \leq & y_2(3) \\ \hline \multicolumn{6}{c}{(y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3} & \leq & y_1 + 3y_2 \end{array}$$

Coefficients

$$\begin{array}{rcrcl} y_1 & + & 3y_2 & \geq & 4 \\ 4y_1 & + & y_2 & \geq & 1 \\ & & y_2 & \geq & 3 \end{array}$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2$ then to attain the best upper bound:

$$\begin{array}{lrcrcl} \min & y_1 & + & 3y_2 & & \\ & y_1 & + & 3y_2 & \geq & 4 \\ & 4y_1 & + & y_2 & \geq & 1 \\ & & & y_2 & \geq & 3 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

Multipliers Approach

$$
\begin{array}{c}
\pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1}
\end{array}
\left[
\begin{array}{cccc|cccc|c|c}
a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\
\vdots & \ddots & & & & & & & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 1 & 0 \\
\hline
c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0
\end{array}
\right]
$$

Working columnwise, since at optimum $\bar{c}_k \leq 0$ for all $k = 1, \dots, n+m$:

$$
\left\{
\begin{array}{ccccccccc}
\pi_1 a_{11} & + & \pi_2 a_{21} & \dots & + & \pi_m a_{m1} & + & \pi_{m+1} c_1 & \leq & 0 \\
\vdots & \ddots & & & & & & & & \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots & + & \pi_m a_{mn} & + & \pi_{m+1} c_n & \leq & 0 \\
\hline
\pi_1 a_{1,n+1}, & & \pi_2 a_{2,n+2}, & \dots & & \pi_m a_{m,n+1} & & & \leq & 0 \\
\hline
 & & & & & & & \pi_{m+1} & = & 1 \\
\hline
\pi_1 b_1 & + & \pi_2 b_2 & \dots & + & \pi_m b_m & & & (\leq & 0)
\end{array}
\right.
$$

(since from the last row $z = -\pi b$ and we want to maximize z then we would $\min(-\pi b)$ or equivalently $\max \pi b$)

$$\begin{array}{llllllll}
\max & \pi_1 b_1 & + & \pi_2 b_2 & \dots & + & \pi_m b_m & & \\
 & \pi_1 a_{11} & + & \pi_2 a_{21} & \dots & + & \pi_m a_{m1} & \leq & -c_1 \\
 & \vdots & \ddots & & & & & & \\
 & \pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots & + & \pi_m a_{mn} & \leq & -c_n \\
 & & & & & & \pi_1, \pi_2, \dots \pi_m & \leq & 0
\end{array}$$

$y = -\pi$

$$\begin{array}{llllllll}
\max & -y_1 b_1 & + & -y_2 b_2 & \dots & + & -y_m b_m & & \\
 & -y_1 a_{11} & + & -y_2 a_{21} & \dots & + & -y_m a_{m1} & \leq & -c_1 \\
 & \vdots & \ddots & & & & & & \\
 & -y_1 a_{1n} & + & -y_2 a_{2n} & \dots & + & -y_m a_{mn} & \leq & -c_n \\
 & & & & & & -y_1, -y_2, \dots - y_m & \leq & 0
\end{array}$$

$$\begin{array}{llll}
\min & w & = & b^T y \\
 & A^T y & \geq & c \\
 & y & \geq & 0
\end{array}$$

Example

$$\begin{array}{lcccccc} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\left\{ \begin{array}{cccccccc} 5\pi_1 & + & 4\pi_2 & + & 6\pi_3 & \leq & 0 \\ 10\pi_1 & + & 4\pi_2 & + & 8\pi_3 & \leq & 0 \\ 1\pi_1 & + & 0\pi_2 & + & 0\pi_3 & \leq & 0 \\ 0\pi_1 & + & 1\pi_2 & + & 0\pi_3 & \leq & 0 \\ 0\pi_1 & + & 0\pi_2 & + & 1\pi_3 & = & 1 \\ 60\pi_1 & + & 40\pi_2 & & & & \end{array} \right.$$

$$\begin{array}{lcccc} y_1 & = & -\pi_1 \geq & 0 \\ y_2 & = & -\pi_2 \geq & 0 \end{array}$$

...

Duality Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\leq$ $\geq$ $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
	$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	jth constraint has $\geq$ $\leq$ $=$

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1. Derivation and Motivation

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Symmetry

The dual of the dual is the primal:

Primal problem:

$$\begin{array}{rrcl} \max & z & = & c^T x \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

Dual Problem:

$$\begin{array}{rrcl} \min & w & = & b^T y \\ & Ay & \geq & c \\ & y & \geq & 0 \end{array}$$

Let's put the dual in the usual form

Dual problem:

$$\begin{array}{rrcl} \min & b^T y & \equiv & -\max -b^T y \\ & -Ay & \leq & -c \\ & y & \geq & 0 \end{array}$$

Dual of Dual:

$$\begin{array}{rrcl} -\min & c^T x & & \\ & -Ax & \geq & -b \\ & x & \geq & 0 \end{array}$$

Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

Theorem (Weak Duality Theorem)

Given:

$$\begin{array}{ll} (P) & \max\{c^T x \mid Ax \leq b, x \geq 0\} \\ (D) & \min\{b^T y \mid A^T y \geq c, y \geq 0\} \end{array}$$

for any feasible solution x *of (P) and any feasible solution* y *of (D):*

$$c^T x \leq b^T y$$

Proof:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \qquad \text{since } c_j \leq \sum_{i=1}^{m} y_i a_{ij} \forall j \text{ and } x_j \geq 0$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_i \right) y_i \leq \sum_{i=1}^{m} b_i y_i$$

Strong Duality Theorem

Theorem (Strong Duality Theorem)

Given:

$$\begin{array}{ll} (P) & \max\{c^T x \mid Ax \le b, x \ge 0\} \\ (D) & \min\{b^T y \mid A^T y \ge c, y \ge 0\} \end{array}$$

exactly one of the following occurs:

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution* $x^* = [x_1^*, \dots, x_n^*]$
   (D) has feasible solution $y^* = [y_1^*, \dots, y_m^*]$

$$c^T x^* = b^T y^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.
- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)
- The last row of the final tableau will give us

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k = z^* + \sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i=1}^{m} \bar{c}_{n+i} x_{n+i} \qquad (*)$$

$$= z^* + \bar{c}_B x_B + \bar{c}_N x_N$$

In addition, $z^* = \sum_{j=1}^{n} c_j x_j^*$ because optimal value

- We define $\boxed{y_i^* = -\bar{c}_{n+i},}\ i = 1, 2, \ldots, m$
- We claim that $(y_1^*, y_2^*, \ldots, y_m^*)$ is a dual feasible solution satisfying $c^T x^* = b^T y^*$.

- Let's verify the claim:
  We substitute in (*) $\sum c_j x_j$ for z and $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, 2, \dots, m$ for slack variables

$$\sum c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

$$= \left(z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

This must hold for every $(x_1, x_2, \dots, x_n)$ hence:

$$z^* = \sum_{i=1}^{m} b_i y_i^* \qquad \implies y^* \text{ satisfies } c^T x^* = b^T y^*$$

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, j = 1, 2, \dots, n$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \ldots, n+m$:

$$\bar{c}_j \leq 0 \rightsquigarrow \quad c_j - \sum_{i=1}^{m} y_i^* a_{ij} \leq 0 \rightsquigarrow \quad \sum_{i=1}^{m} y_i^* a_{ij} \geq c_j \qquad j = 1, 2, \ldots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow \quad y_i^* = -\hat{c}_{n+i} \geq 0, \qquad i = 1, 2, \ldots, m$$

$\Longrightarrow$ y^* is also dual feasible solution

Complementary Slackness Theorem

Theorem (Complementary Slackness)

A feasible solution x^* for (P)
A feasible solution y^* for (D)
Necessary and sufficient conditions for optimality of both:

$$\left(c_j - \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j^* = 0, \quad j = 1, \ldots, n$$

If $x_j^* \neq 0$ then $\sum y_i^* a_{ij} = c_j$ (no surplus)
If $\sum y_i^* a_{ij} > c_j$ then $x_j^* = 0$

Proof:

$$z^* = cx^* \leq y^* Ax^* \leq by^* = w^*$$

Hence from strong duality theorem:

$$cx^* - yAx^* = 0$$

In scalars

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be $= 0$

Dual Simplex

Dual simplex (Lemke, 1954): apply the simplex method to the dual problem and observe what happens in the primal tableaux:

- Primal works with feasible solutions towards optimality
- Dual works with optimal solutions towards feasibility

Primal simplex on primal problem:

1. pivot > 0
2. col c_j with wrong sign
3. row: $\min \left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, .., m \right\}$

Dual simplex on primal problem:

1. pivot < 0
2. row $b_i < 0$ (condition of feasibility)
3. col: $\min \left\{ \left| \frac{c_j}{a_{ij}} \right| : a_{ij} > 0, j = 1, 2, .., n + m \right\}$ (least worsening solution)

It can work better in some cases than the primal.
Eg. since running time in practice between $2m$ and $3m$, then if $m = 99$ and $n = 9$ then better the dual
Dual based Phase I algorithm (Dual-primal algorithm) (see Sheet 3)

Dual Simplex
Example

Primal:

$$
\begin{array}{lrcrcr}
\max & -x_1 & - & x_2 & & \\
 & -2x_1 & - & x_2 & \leq & 4 \\
 & -2x_1 & + & 4x_2 & \leq & -8 \\
 & -x_1 & + & 3x_2 & \leq & -7 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

- Initial tableau

```
|   | x1 | x2 | w1 | w2 | w3 | -z |  b |
|---+----+----+----+----+----+----+----|
|   | -2 | -1 |  1 |  0 |  0 |  0 |  4 |
|   | -2 |  4 |  0 |  1 |  0 |  0 | -8 |
|   | -1 |  3 |  0 |  0 |  1 |  0 | -7 |
|---+----+----+----+----+----+----+----|
|   | -1 | -1 |  0 |  0 |  0 |  1 |  0 |
```

infeasible start

- x_1 enters, w_2 leaves

Dual:

$$
\begin{array}{lrcrcrcr}
\min & 4y_1 & - & 8y_2 & - & 7y_3 & & \\
 & -2y_1 & - & 2y_2 & - & y_3 & \geq & -1 \\
 & -y_1 & + & 4y_2 & + & 3y_3 & \geq & -1 \\
 & & & & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

- Initial tableau ($\min by \equiv -\max -by$)

```
|   | y1 | y2 | y3 | z1 | z2 | -p | b |
|---+----+----+----+----+----+----+---|
|   |  2 |  2 |  1 |  1 |  0 |  0 | 1 |
|   |  1 | -4 | -3 |  0 |  1 |  0 | 1 |
|---+----+----+----+----+----+----+---|
|   | -4 |  8 |  7 |  0 |  0 |  1 | 0 |
```

feasible start (thanks to $-x_1 - x_2$)

- y_2 enters, z_1 leaves

- x_1 enters, w_2 leaves

```
|   | x1 | x2 | w1 |   w2 | w3 | -z |  b |
|---+----+----+----+------+----+----+----|
|   |  0 | -5 |  1 |   -1 |  0 |  0 | 12 |
|   |  1 | -2 |  0 | -0.5 |  0 |  0 |  4 |
|   |  0 |  1 |  0 | -0.5 |  1 |  0 | -3 |
|---+----+----+----+------+----+----+----|
|   |  0 | -3 |  0 | -0.5 |  0 |  1 |  4 |
```

- w_2 enters, w_3 leaves (note that we kept $c_j > 0$, ie, optimality)

```
|   | x1 | x2 | w1 | w2 | w3 | -z |  b |
|---+----+----+----+----+----+----+----|
|   |  0 | -7 |  1 |  0 | -2 |  0 | 18 |
|   |  1 | -3 |  0 |  0 | -1 |  0 |  7 |
|   |  0 | -2 |  0 |  1 | -2 |  0 |  6 |
|---+----+----+----+----+----+----+----|
|   |  0 | -4 |  0 |  0 | -1 |  1 |  7 |
```

- y_2 enters, z_1 leaves

```
|   | y1 | y2 |  y3 |  z1 | z2 | -p |   b |
|---+----+----+-----+-----+----+----+-----|
|   |  1 |  1 | 0.5 | 0.5 |  0 |  0 | 0.5 |
|   |  5 |  0 |  -1 |   2 |  1 |  0 |   3 |
|---+----+----+-----+-----+----+----+-----|
|   | -4 |  0 |   3 | -12 |  0 |  1 |  -4 |
```

- y_3 enters, y_2 leaves

```
|   |  y1 | y2 | y3 | z1 | z2 | -p |  b |
|---+-----+----+----+----+----+----+----|
|   |   2 |  2 |  1 |  1 |  0 |  0 |  1 |
|   |   7 |  2 |  0 |  3 |  1 |  0 |  3 |
|---+-----+----+----+----+----+----+----|
|   | -18 | -6 |  0 | -7 |  0 |  1 | -7 |
```

Economic Interpretation

$$
\begin{array}{lccccccc}
\max & 5x_0 & + & 6x_1 & + & 8x_2 & & \\
 & 6x_0 & + & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 8x_0 & + & 4x_1 & + & 4x_2 & \geq & 40 \\
 & 4x_0 & + & 5x_1 & + & 6x_2 & \geq & 50 \\
 & & & & & x_0, x_1, x_2 & \geq & 0
\end{array}
$$

final tableau:

$x0$	$x1$	$x2$	$s1$	$s2$	$s3$	$-z$	b
	0	1		0			5/2
	1	0		0			7
	0	0		1			2
$-1/5$	0	0	$-1/5$	0	-1		62

- Which are the values of variables, the reduced costs, the shadow prices (or marginal price), the values of dual variables?
- If one slack variable > 0 then overcapacity
- How many products can be produced at most? at most m
- How much more expensive a product not selected should be? look at reduced costs: $c - \pi A > 0$
- What is the value of extra capacity of manpower? In 1+1 out 1/5+1

Game: Suppose two economic operators:

- P owns the factory and produces goods
- D is the market buying and selling raw material and resources
- D asks P to close and sell him all resources
- P considers if the offer is convenient
- D wants to spend less possible
- y are prices that D offers for the resources
- $\sum y_i b_i$ is the amount D has to pay to have all resources of P
- $\sum y_i a_{ij} \geq c_j$ total value to make j > price per unit of product
- P either sells all resources $\sum y_i a_{ij}$ or produces product j (c_j)
- without $\geq$ there would not be negotiation because P would be better off producing and selling
- at optimality the situation is indifferent (strong th.)
- resource 2 that was not totally utilized in the primal has been given value 0 in the dual. (complementary slackness th.) Plausible, since we do not use all the resource, likely to place not so much value on it.
- for product 0 $\sum y_i a_{ij} > c_j$ hence not profitable producing it. (complementary slackness th.)

Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian (to do)
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
- Dual Simplex
- Economic interpretation