DM545

Linear and Integer Programming

Lecture 5
Sensitivity Analysis
Revised Simplex Method

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Outline

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Lagrangian Duality

Relaxation: if a problem is hard to solve then find an easier problem resembling the original one that provides information in terms of bounds. Then search strongest bounds.

$$
\begin{aligned}
\min\ 13x_1 + 6x_2 + 4x_3 + 12x_4 & \\
2x_1 + 3x_2 + 4x_3 + 5x_4 &= 7 \\
3x_1 + + 2x_3 + 4x_4 &= 2 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

We wish to reduce to a problem easier to solve, ie:

$$
\begin{aligned}
\min\ c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\
x_1, x_2, \ldots, x_n &\geq 0
\end{aligned}
$$

solvable by inspection: if $c < 0$ then $x = +\infty$, if $c \geq 0$ then $x = 0$.
measure of violation of the constraints:

$$
\begin{aligned}
&7 - (2x_1 + 3x_2 + 4x_3 + 5x_4) \\
&2 - (3x_1 + + 2x_3 + 4x_4)
\end{aligned}
$$

We relax these measures in the obj. function with Lagrangian multipliers y_1, y_2.
We obtain a family of problems:

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{rrrrr} & 13x_1 & + 6x_2 & + 4x_3 & + 12x_4 \\ +y_1(7- & 2x_1 & + 3x_2 & + 4x_3 & + 5x_4) \\ +y_2(2- & 3x_1 & + & + 2x_3 & + 4x_4) \end{array} \right\}$$

1. for all $y_1, y_2 \in \mathbb{R} : \mathrm{opt}(PR(y_1, y_2)) \leq \mathrm{opt}(P)$
2. $\max_{y_1, y_2 \in \mathbb{R}} \{\mathrm{opt}(PR(y_1, y_2))\} \leq \mathrm{opt}(P)$

PR is easy to solve.
(It can be also seen as a proof of the weak duality theorem)

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{rl} & (13 - 2y_2 - 3y_2)\, x_1 \\ + & (6 - 3y_1 \qquad\quad)\, x_2 \\ + & (4 \qquad\quad - 2y_2)\, x_3 \\ + & (12 - 5y_1 - 4y_2)\, x_4 \\ + & \qquad 7y_1 + 2y_2 \end{array} \right\}$$

if coeff. of x is < 0 then bound is $-\infty$ then LB is useless

$$\begin{array}{r} (13 - 2y_2 - 3y_2) \geq 0 \\ (6 - 3y_1 \qquad\quad) \geq 0 \\ (4 \qquad\quad - 2y_2) \geq 0 \\ (12 - 5y_1 - 4y_2) \geq 0 \end{array}$$

If they all hold then we are left with $7y_1 + 2y_2$ because all go to 0.

$$\begin{array}{rl} \max 7y_1 + 2y_2 & \\ 2y_2 + 3y_2 & \leq 13 \\ 3y_1 \qquad\quad & \leq 6 \\ + 2y_2 & \leq 4 \\ 5y_1 + 4y_2 & \leq 12 \end{array}$$

General Formulation

$$\begin{aligned} \max \quad & z = c^T x & c \in \mathbb{R}^n \\ & Ax = b & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\ & x \geq 0 & x \in \mathbb{R}^n \end{aligned}$$

$$\max_{y \in \mathbb{R}^m} \{ \min_{x \in \mathbb{R}^n_+} \{ cx + y(b - Ax) \} \}$$

$$\max_{y \in \mathbb{R}^m} \{ \min_{x \in \mathbb{R}^n_+} \{ (c - yA)x + yb \} \}$$

$$\begin{aligned} \max \quad & b^T y \\ & A^T y \leq c \\ & y \in \mathbb{R}^m \end{aligned}$$

Outline

Sensitivity Analysis

aka Postoptimality Analysis

Instead of solving each of the modified problems from scratch, exploit results obtained from solving the original problem.

$$\max\{c^T x \mid Ax = b, l \leq x \leq u\} \qquad (*)$$

(I) changes to coefficients of objective function:
$\max\{\tilde{c}^T x \mid Ax = b, l \leq x \leq u\}$ (primal)
x^* of (*) remains feasible hence we can restart the simplex from x^*

(II) changes to RHS terms: $\max\{c^T x \mid Ax = \tilde{b}, l \leq x \leq u\}$ (dual)
x^* optimal feasible solution of (*)
basic sol $\bar{x}$ of (II): $\bar{x}_N = x^*$, $A_B\bar{x}_B = \tilde{b} - A_N\bar{x}_N$
$\bar{x}$ is dual feasible and we can start the dual simplex from there. If $\tilde{b}$ differs from b only slightly it may be we are already optimal.

(III) introduce a new variable: (primal)

$$\begin{aligned} \max \quad & \sum_{j=1}^{6} c_j x_j \\ & \sum_{j=1}^{6} a_{ij} x_j = b_i, \ i = 1, \dots, 3 \\ & l_j \le x_j \le u_j, \ j = 1, \dots, 6 \\ & [x_1^*, \dots, x_6^*] \text{ feasible} \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^{7} c_j x_j \\ & \sum_{j=1}^{7} a_{ij} x_j = b_i, \ i = 1, \dots, 3 \\ & l_j \le x_j \le u_j, \ j = 1, \dots, 7 \\ & [x_1^*, \dots, x_6^*, 0] \text{ feasible} \end{aligned}$$

(IV) introduce a new constraint: (dual)

$$\sum_{j=1}^{6} a_{4j} x_j = b_4$$

$$\sum_{j=1}^{6} a_{5j} x_j = b_5$$

$$l_j \le x_j \le u_j \qquad j = 7, 8$$

$$[x_1^*, \dots, x_6^*] \text{ optimal}$$

$$[x_1^*, \dots, x_6^*, x_7^*, x_8^*] \text{ feasible}$$

$$x_7^* = b_4 - \sum_{j=1}^{6} a_{4j} x_j^*$$

$$x_8^* = b_5 - \sum_{j=1}^{6} a_{5j} x_j^*$$

Examples

(I) Variation of reduced costs:

$$\begin{aligned} \max\ & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

The last tableau gives the possibility to estimate the effect of variations

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
x_1	1	0	−1/5	1/2	0	8
	0	0	−2/5	−1	1	−64

For a variable in basis

$$\max(6+\delta)x_1 + 8x_2$$

the perturbation goes unchanged in the red. costs.:

$$-\frac{2}{5}\cdot 5 - 1\cdot 4 + 1(6+\delta)$$

For a variable not in basis it may change the sign of the reduced cost $\Longrightarrow$ worth bringing in basis $\Longrightarrow$the δ term propagates to other columns

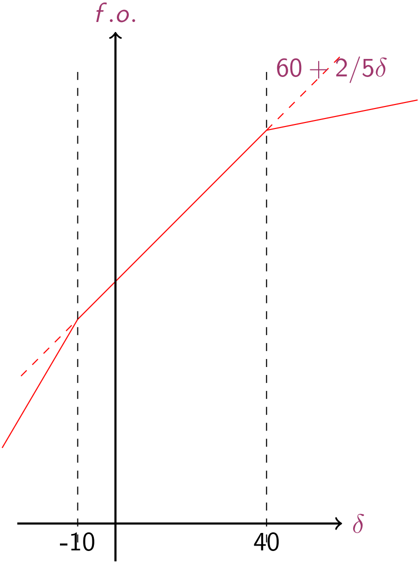
(II) Changes in RHS terms

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	$60+\delta$
x_4	4	4	0	1	0	$40+\epsilon$
	6	8	0	0	1	0

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	$2+1/5\delta-1/4\epsilon$
x_1	1	0	$-1/5$	$1/2$	0	$8-1/5\delta+1/2\epsilon$
	0	0	$-2/5$	-1	1	$-64-2/5\delta-\epsilon$

It would be more convenient to augment the second.
If $60+\delta \Longrightarrow$all RHS terms change and we must check feasibility
Which are the multipliers for the first row?$k_1=\frac{1}{5}, k_2=-\frac{1}{4}, k_3=0$
I: $1/5(60+\delta)-1/4\cdot 40+0\cdot 0=12+\delta/5-10=2+\delta/5$
II: $-1/5(60+\delta)+1/2\cdot 40+0\cdot 0=-60/5+20-\delta/5=8-1/5\delta$
Risk that RHS becomes negative
Eg: if $\delta=-20 \Longrightarrow$tableau stays optimal but not feasible $\Longrightarrow$apply dual simplex

Graphical Representation



(III) Add a variable

$$\begin{aligned} \max\ 5x_0 + 6x_1 + \ 8x_2 & \\ 6x_0 + 5x_1 + 10x_2 &\leq 60 \\ 8x_0 + 4x_1 + \ 4x_2 &\leq 40 \\ x_0, x_1, x_2 &\geq \ 0 \end{aligned}$$

Reduced cost of x_0? $c_j - \sum \pi_i a_{ij} = -\frac{2}{5} + (-1)8 + 1 \cdot 5 = -\frac{27}{5}$

To make worth entering in basis:

- increase its cost
- decrease the amount in constraint II: $-2/5 \cdot 6 - a_{20} + 5 > 0$

(IV) Add a constraint

$$
\begin{aligned}
\max\ 6x_1 + \ 8x_2 & \\
5x_1 + 10x_2 &\leq 60 \\
4x_1 + \ 4x_2 &\leq 40 \\
5x_1 + \ 6x_2 &\leq 50 \\
x_1, x_2 &\geq \ 0
\end{aligned}
$$

Final tableau not in canonical form, need to iterate

	x_1	x_2	x_3	x_4	x_5	$-z$	b
x_2	0	1	$1/5$	$-1/4$		0	2
x_1	1	0	$-1/5$	$1/2$		0	8
	0	0	$5/5$	$6/4$	1	0	-2
	0	0	$-2/5$	-1	0	1	-64

(V) change in a technological coefficient:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	$10+\delta$	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

- first effect on its column
- then look at c
- finally look at b

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	$(10+\delta)1/5+4(-1/4)$	$1/5$	$-1/4$	0	2
x_1	1	$(10+\delta)(-1/5)+4(1/2)$	$-1/5$	$1/2$	0	8
	0	$-2/5\delta$	$-2/5$	-1	1	-64

Resume

Advantages of considering the dual formulation:

- proving optimality (although the simplex tableau can already do that)
- gives a way to check the correctness of results easily
- alternative solution method (ie, primal simplex on dual)
- sensitivity analysis
- solving P or D we solve the other for free
- certificate of infeasibility