DM545

Linear and Integer Programming

Lecture 7

Integer Linear Programming Modeling

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Outline

Discrete Optimization

- Often need to deal with integral inseparable quantities.
- Sometimes rounding can go.
- Other times rounding not feasible: eg, presence of a bus on a line is 0.3...

Integer Linear Programming

Linear Objective
Linear Constraints
but! integer variables

The world is not linear: OR is the art and science of obtaining bad answers to questions to which otherwise worse answers would be given

$$\begin{aligned} \max\ & c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Linear Programming (LP)

$$\begin{aligned} \max\ & c^T x + h^T y \\ Ax + \quad Gy &\leq b \\ x &\geq 0 \\ y &\geq 0 \text{ and integer} \end{aligned}$$

(Linear) Mixed Integer Programming (MIP)

$$\begin{aligned} \max\ & c^T x \\ Ax &\leq b \\ x &\geq 0 \text{ and integer} \end{aligned}$$

Integer (Linear) Programming

$$\begin{aligned} \max\ & c^T x \\ Ax &\leq b \\ x &\in \{0,1\}^n \end{aligned}$$

Binary Integer Program (BIP)
0/1 Integer Programming

$$\begin{aligned} \max\ & f(x) \\ g(x) &\leq b \\ x &\geq 0 \end{aligned}$$

Non-linear Programming (NLP)

Combinatorial Optimization Problems

Definition (Combinatorial Optimization Problem (COP))

Given: Finite set $N = \{1, \ldots, n\}$ of objects, weights $c_j \forall j \in N$, $\mathcal{F}$ a collection of feasible subsets of N
Find a minimum weight feasible subset:

$$\min_{S \subseteq N} \left\{ \sum_{j \in S} c_j \mid S \in \mathcal{F} \right\}$$

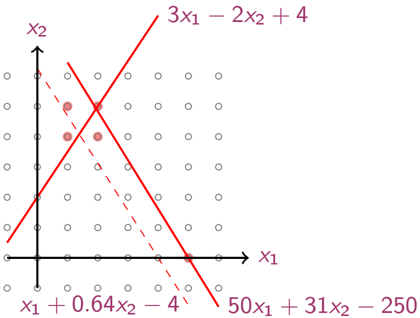
Many COP can be modelled as IP or BIP.

Typically: incidence vector of S, $x^S \in \mathbb{R}^n$: $x_j^S = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}$

Rounding

$$\begin{aligned} \max\ & 100x_1 + 64x_2 \\ & 50x_1 + 31x_2 \leq 250 \\ & 3x_1 - \ \ 2x_2 \geq -4 \\ & x_1, x_2 \in \mathbb{Z}^+ \end{aligned}$$

LP optimum $(376/193, 950/193)$
IP optimum $(5, 0)$



$\rightsquigarrow$ feasible region convex but not continuous: Now the optimum can be on the border (vertices) but also internal.
Possible way: solve the relaxed problem.

- If solution is integer, done.
- If solution is rational (never irrational) try rounding to the nearest integers (but may exit feasibility region)
  if in $\mathbb{R}^2$ then 2^2 possible roundings (up or down)
  if in $\mathbb{R}^n$ then 2^n possible roundings (up or down)

Note: rounding does not help in the example above

Rational Solutions

Theorem

Optimal feasible solutions to LP problems are always rational as long as all coefficient and constants are rational.

Proof: derives from the fact that in the simplex we only perform multiplications, divisions and sums of rational numbers

BIP Modeling

Binary integer programming allows to model alternative choices:

- Eg: 2 feasible regions, ie, disjunctive constraints, not possible in LP. introduce y auxiliary binary variable and M a big number:

$$\begin{aligned} Ax &\leq b + My && \text{if } y = 0 \text{ then this is active} \\ A'x &\leq b' + M(1-y) && \text{if } y = 1 \text{ then this is active} \end{aligned}$$

- at least $h \leq k$ of $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, \ldots, k$ must be satisfied
  introduce y_i, $i = 1, ..., k$ auxiliary binary variables

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + My_i$$

$$\sum_{i} y_i \leq k - h$$

Outline

How to Build a Model

- **First:** Find out exactly what the decision maker needs to know
  - No point finding the right solution to the wrong problem ...
  - Which job j should person i do?
  - Where to locate your storage facility, which customers will receive goods from which facility, and how much?
- **So:** Define **Decision Variables** of suitable type (continuous, integer valued, binary) corresponding to the needs
- **Then:** Formulate an **Objective Function** computing the benefit/cost
- **Finally:** Formulate mathematical **constraints** indicating the interplay between the different variables.

How to “build” a constraint

- Formulate relationship between the variables in plain words
- Then formulate your sentences using logical connectives **and,or,not, implies**
- Finally convert the logical statement to a mathematical constraint.

Example

- “The power plant must not work in two neighbouring time periods”
- on/off is modelled using **binary** integer variables
- $x_i = 1$ implies $\Rightarrow x_{i+1} = 0$
- $x_i + x_{i+1} \leq 1$

Outline

The Assignment Problem

The problem ..

The assignment problem is a well known optimization problem where **assignees** are being assigned to perform **tasks**. Assigning people to jobs is a common application of the assignment problem.

Suppose we have n people and n jobs, and that each person has a certain proficiency at each job.

Formulate a mathematical model that can be used to find an assignment that maximizes the total proficiency.

The Assignment Problem

Decision Variables:

$$x_{ij} = \begin{cases} 1 \text{ if person } i \text{ is assigned job } j, \text{ for } i, j = 1, 2, \ldots, n \\ 0 \text{ otherwise,} \end{cases}$$

Objective Function:

$$\max \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} x_{ij}$$

where ρ_{ij} is person i's proficiency at job j

The Assignment Problem

Constraints:

Each person is assigned one job:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i$$

e.g. for person one we get $x_{11} + x_{12} + x_{13} + \cdots + x_{1n} = 1$

Each job is assigned to one person:

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for all } j$$

e.g. for job one we get $x_{11} + x_{21} + x_{31} + \cdots + x_{n1} = 1$

Outline

Set Covering

Given: a number of regions, a number of centers, regions that can be served in less than 8 minutes, cost of installing an emergency center.
Task: Where to install a set of emergency centers such that the total cost is minimized and all regions safely served?

As a COP: $M = \{1, \ldots, m\}$ regions, $N\{1, \ldots, n\}$ centers, $S_j \subseteq M$ regions serviced by j

$$\min_{T \in N} \left\{ \sum_{j \in T} c_j : \cup_{j \in T} S_j = M \right\}$$

As a BIP:

Variables:
$x \in \mathbb{B}^n$, $x_j = 1$ if center j is selected, 0 otherwise
Objective:

$$\min \sum_{j=1}^{n} c_j x_j$$

Constraints:

- incidence matrix: $a_{ij} = \begin{cases} 1 \\ 0 \end{cases}$
- $\sum_{j=1}^{n} a_{ij} x_j \geq 1$

Example

- $M = \{1, \ldots, 5\}, N = \{1, \ldots, 6\}, c_j = 1 \forall j = 1, \ldots, n$
  $S_1 = (1, 2), S_2 = (1, 3, 5), S_3 = (2, 4, 5), S_4 = (3), S_5 = (1), S_6 = (4, 5)$
-
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Set covering:

1. min, $\geq$
2. all RHS terms are 1
3. all matrix elements are 1

$$\begin{aligned} \min\ & c^T x \\ & Ax \geq 1 \\ & x \in \mathbb{B}^n \end{aligned}$$

Set packing
select as many of M without overlap

1. max, $\leq$
2. all RHS terms are 1
3. all matrix elements are 1

$$\begin{aligned} \max\ & c^T x \\ & Ax \leq 1 \\ & x \in \mathbb{B}^n \end{aligned}$$

Set partitioning
cover exactly once each element of M

1. max or min, $=$
2. all RHS terms are 1
3. all matrix elements are 1

$$\begin{aligned} \max\ & c^T x \\ & Ax = 1 \\ & x \in \mathbb{B}^n \end{aligned}$$

Generalization: $RHS \geq 1$
Other application examples:

- Aircrew scheduling: M: legs to cover, N: rosters
- Vehicle routing: M: customers, N: routes

Manpower Planning

- Each person covers 7 hours
- A person starting in hour 3 contributes to the workload in hours 3,4,5,6,7,8,9
- A person starting in hour i contributes to the workload in hours $i, \ldots, i+6$

Modelling task

Formulate a mathematical model to determine the number of people required to cover the workload

Decision Variables:

- $x_i \in \mathbb{N}_0$: number of people starting work in hour $i (i = 1, \ldots, 15)$

Objective Function:

$$\min \sum_{i=1}^{9} x_i$$

Constraints:

- Demand:

$$\sum_{i=t-6}^{i=t} x_i \geq d_t \text{ for } t = 1, \ldots, 15$$

- Bounds:

$$x_{-5}, \ldots, x_0 = 0$$

A good written example:

2.1. Notation

Let N be the set of operational flight legs and K the set of aircraft types. Denote by n^k the number of available aircraft of type $k \in K$. Define Ω^k, indexed by p, as the set of feasible schedules for aircraft of type $k \in K$ and let index $p = 0$ denote the empty schedule for an aircraft. Next associate with each schedule $p \in \Omega^k$ the value c_p^k denoting the anticipated profit if this schedule is assigned to an aircraft of type $k \in K$ and a_{ip}^k a binary constant equal to 1 if this schedule covers flight leg $i \in N$ and 0 otherwise. Furthermore, let S be the set of stations and $S^k \subseteq S$ the subset having the facilities to serve aircraft of type $k \in K$. Then, define o_{sp}^k and d_{sp}^k to equal to 1 if schedule p, $p \in \Omega^k$, starts and ends respectively at station s, $s \in S^k$, and 0 otherwise.

Denote by θ_p^k, $p \in \Omega^k \backslash \{0\}$, $k \in K$, the binary decision variable which takes the value 1 if schedule p is assigned to an aircraft of type k, and 0 otherwise. Finally, let θ_0^k, $k \in K$, be a nonnegative integer variable which gives the number of unused aircraft of type k.

2.2. Formulation

Using these definitions, the DARSP can be formulated as:

$$\text{Maximize} \sum_{k \in K} \sum_{p \in \Omega^k} c_p^k \theta_p^k \qquad (1)$$

subject to:

$$\sum_{k \in K} \sum_{p \in \Omega^k} a_{ip}^k \theta_p^k = 1 \quad \forall i \in N, \qquad (2)$$

$$\sum_{p \in \Omega^k} (d_{sp}^k - o_{sp}^k)\theta_p^k = 0 \quad \forall k \in K, \forall s \in S^k, \qquad (3)$$

$$\sum_{p \in \Omega^k} \theta_p^k = n^k \quad \forall k \in K, \qquad (4)$$

$$\theta_p^k \geq 0 \quad \forall k \in K, \forall p \in \Omega^k, \qquad (5)$$

$$\theta_p^k \text{ integer} \quad \forall k \in K, \forall p \in \Omega^k. \qquad (6)$$

The objective function (1) states that we wish to maximize the total anticipated profit. Constraints (2) require that each operational flight leg be covered exactly once. Constraints (3) correspond to the flow conservation constraints at the beginning and the end of the day at each station and for each aircraft type. Constraints (4) limit the number of aircraft of type $k \in K$ that can be used to the number available. Finally, constraints (5) and (6) state that the decision variables are nonnegative integers. This model is a Set Partitioning problem with additional constraints.

[from G. Desaulniers, J. Desrosiers, Y. Dumas, M.M. Solomon and F. Soumis. Daily Aircraft Routing and Scheduling. Management Science, 1997, 43(6), 841-855]

Outline

Matching

Definition (Matching Theory Terminology)

Matching: set of pairwise non adjacent edges

Covered (vertex): a vertex is covered by a matching M if it is incident to an e in M

Perfect (matching): if M covers each vertex in G

Maximum (matching): if M covers as many vertices as possible

Matchable (graph): if the graph G has a perfect matching

$$
\begin{aligned}
\max \quad & \sum_{v \in V} w_e x_e \\
& \sum_{e \in E : v \in e} x_e = 1 \qquad \forall v \in V \\
& x_e = \{0, 1\} \ \forall e \in E
\end{aligned}
$$

Special case: bipartite matching $\equiv$ assignment problems

Vertex Cover

Select a subset $S \subseteq V$ such that each edge has at least one end vertex in S.

$$
\begin{aligned}
\min \quad & \sum_{v \in V} x_v \\
x_v + x_u \geq 1 \quad & \forall u, v \in V, uv \in E \\
x_v = \{0, 1\} \quad & \forall v \in V
\end{aligned}
$$

The LP relaxation gives a 2-approximation

Maximum independent Set

Find the largest subset $S \subseteq V$ such that the induced graph has no edges

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ x_v + x_u &\leq 1 \qquad \forall u, v \in V, uv \in E \\ x_v &= \{0, 1\} \; \forall v \in V \end{aligned}$$

LP relaxation gives an $O(n)$-approximation (almost useless)

Travelling Salesman Problem

- Find the cheapest movement for a stacker crane that must pick up and drop objects

- n cities, c_{ij} cost of travel

Variables:
Objective: