DM545

Linear and Integer Programming

Lecture 9
Well Solved Problems
Network Flows

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Well Solved Problems

2. (Minimum Cost) Network Flows

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1. Well Solved Problems

2. (Minimum Cost) Network Flows

Theoretical analysis to prove results about

- strength of certain inequalities that are facet defining
  2 ways
- descriptions of convex hull of some discrete $X \subseteq \mathbb{Z}^*$
  several ways, we see one next

Example

Example: Let $X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^m \leq my, x_i \leq 1 \text{ for } i = 1, \ldots, m$
and $P = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$.
Polyhedron P describes $\text{conv}(X)$

Totally Unimodular Matrices

When the LP solution to this problem

$$IP : \max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n_+\}$$

with all data integer will have integer solution?

$$\left[\begin{array}{c|c|c|c} A & I & 0 & b \\ \hline c & 0 & 1 & 0 \end{array}\right]$$

$A_B x_B + A_N x_N = b$
$A_B x_B = b$, A_B
$m \times m$ non singular matrix

Cramer's rule for solving systems of equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad x = B^{-1} b = \frac{B^{adj} b}{\det(B)}$$

Definition

- A square integer matrix B is called unimodular (UM) if $\det(B) = \pm 1$
- An integer matrix A is called totally unimodular (TUM) if every square, nonsingular submatrix of A is UM

Proposition

- *If A is TUM then all vertices of $R_1(A) = \{x : Ax = b, x \geq 0\}$ are integer if b is integer*
- *If A is TUM then all vertices of $R_2(A) = \{x : Ax \leq b, x \geq 0\}$ are integer if b is integer.*

Proof: if A is TUM then $\left[A \,|\, I \right]$ is TUM
Any square, nonsingular submatrix C of $\left[A \,|\, I \right]$ can be written as

$$C = \left[\begin{array}{c|c} B & 0 \\ \hline D & I_k \end{array} \right]$$

where B is square submatrix of A. Hence $\det(C) = \det(B) = \pm 1$

Proposition

The transpose matrix A^T of a TUM matrix A is also TUM.

Theorem (Sufficient condition)

An integer matrix A with is TUM if

1. $a_{ij} \in \{0, -1, +1\}$ *for all* i, j
2. *each column contains at most two non-zero coefficients* ($\sum_{i=1}^{m} |a_{ij}| \leq 2$)
3. *if the rows can be partitioned into two sets* I_1, I_2 *such that:*
   - *if a column has 2 entries of same sign, their rows are in different sets*
   - *if a column has 2 entries of different signs, their rows are in the same set*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: by induction

Basis: one matrix of one element is TUM

Induction: let C be of size k.
If C has column with all 0s then it is singular.
If a column with only one 1 then expand on that by induction
If 2 non-zero in each column then

$$\forall j : \sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij}$$

but then linear combination of rows and $\det(C) = 0$

Note:
For TUM matrices 2., 3. and 4. hold. 1. also holds: Algorithm to test this in polynomial time due to Seymour

Other matrices with integrality property:

- TUM
- Balanced matrices
- Perfect matrices
- Integer vertices

Defined in terms of forbidden substructures that represent fractionating possibilities.

Proposition

A is always TUM if it comes from

- *node-edge incidence matrix of undirected bipartite graphs (ie, no odd cycles)* ($I_1 = U, I_2 = V, B = (U, V, E)$)
- *node-arc incidence matrix of directed graphs* ($I_2 = \emptyset$)

Eg: Shortest path, max flow, min cost flow, bipartite weighted matching

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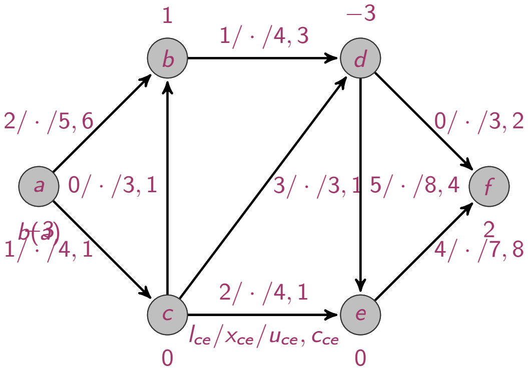
2. (Minimum Cost) Network Flows

Terminology

Network:
- directed graph $D = (V, A)$
- arc, directed link, from tail to head
- lower bound $l_{ij} > 0, \forall ij \in A$, capacity $u_{ij} \geq l_{ij}, \forall ij \in A$
- cost c_{ij}, linear variation (if $ij \notin A$ then $l_{ij} = u_{ij} = 0, c_{ij} = 0$)
- balance vector $b(i)$, $b(i) < 0$ supply node, $b(i) > 0$ demand node, $b(i) = 0$ transhipment node (assumption $\sum_i b(i) = 0$)

$N = (V, A, l, u, b, c)$



Network Flows

Flow $x : A \to \mathbb{R}$

balance vector of x: $b_x(v) = \sum_{vw \in A} x_{vw} - \sum_{uv \in A} x_{uv}, \forall v \in V$

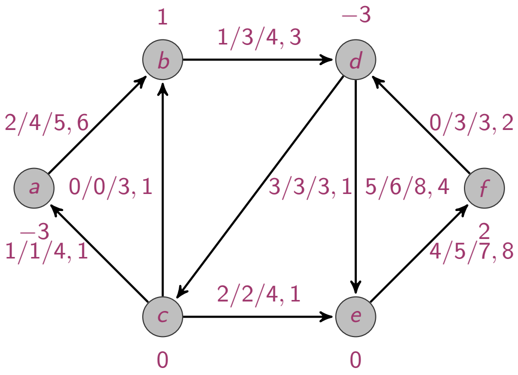
$$b_x(v) \begin{cases} > 0 & \text{sink/target/tank} \\ < 0 & \text{source} \\ = 0 & \text{balanced} \end{cases}$$

(generalizes the concept of path with $b_x(v) = \{0, 1, -1\}$)

feasible	$l_{ij} \leq x_{ij} \leq u_{ij}$, $b_x(i) = b(i)$
cost	$c^T x = \sum_{ij \in A} c_{ij} x_{ij}$ (varies linearly with x)

If iji is a 2-cycle and all $l_{ij} = 0$, then at least one of x_{ij} and x_{ji} is zero.

Example



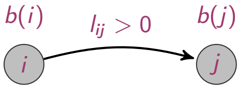
Feasible flow of cost 109

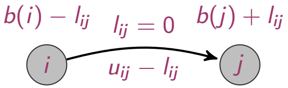
Reductions/Transformations

Lower bounds

Let $N = (V, A, l, u, b, c)$

$$
\begin{aligned}
N' &= (V, A, l', u', b') \\
b'(j) &= b(j) + l_{ij} \\
b'(i) &= b(i) - l_{ij} \\
u'_{ij} &= u_{ij} - l_{ij} \\
l'_{ij} &= 0
\end{aligned}
$$



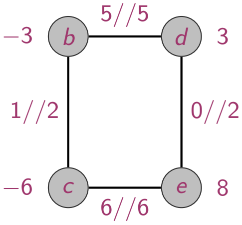


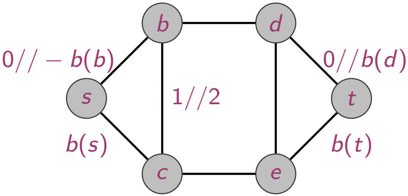
$$c^T x$$

$$c^T x' + \sum_{ij \in A} c_{ij} l_{ij}$$

(s, t)-**flow:**

$$b_x(v) = \begin{cases} -k & \text{if } v = s \\ k & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}, \qquad |x| = b_x(s)$$

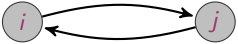




$$b(s) = \sum_{v:v(v)<0} b(v) = -M$$
$$b(t) = \sum_{v:v(v)>0} b(v) = M$$

$\exists$ feasible flow in $N \iff \exists$ (s, t)-flow in N_{st} with $|x| = M \iff$ max flow in N_{st} is M

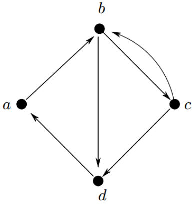
Undirected arcs

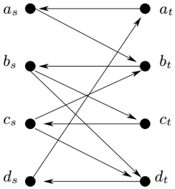


Vertex splitting

If there are bounds and costs of flow passing thorugh vertices where $b(v) = 0$ (used to ensure that a node is visited):

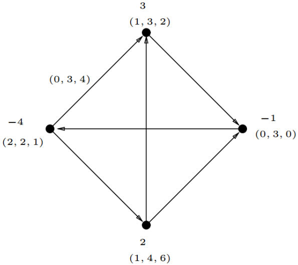
$N = (V, A, l, u, c, l^*, u^*, c^*)$

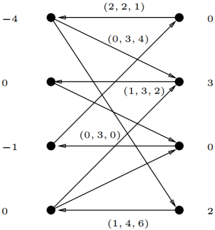




From D to D_{ST} as follows:

$$\forall v \in V \quad \rightsquigarrow v_s, v_t \in V(D_{ST}) \text{ and } v_t v_s \in A(D_{ST})$$
$$\forall xy \in A(D) \rightsquigarrow x_s y_t \in A(D_{ST})$$





$$\forall xy \in A \text{ and } x_s y_t \in A_{ST} \rightsquigarrow h'(x_s y_t) = h(x, y) \;\; h \in \{l, u, c\}$$
$$\forall v \in V \text{ and } v_t v_s \in A_{ST} \rightsquigarrow h'(v_t, v_s) = h^*(v) \;\; h^* \in \{l^*, u^*, c^*\}$$

If $b(v) = 0$, then $b'(v_s) = b'(v_t) = 0$
If $b(v) > 0$, then $b'(v_t) = b(v)$ and $b'(v_s) = 0$
If $b(v) < 0$, then $b'(v_t) = 0$ and $b'(v_s) = b(v)$

Minimum Cost Network Flows

Find cheapest flow through a network in order to satisfy demands at certain nodes from available supplier nodes.

Variables:

$$x_{ij}$$

Objective:

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\begin{aligned} \min\ & c^T x \\ & Nx = b \\ & 0 \leq x \leq u \end{aligned}$$

Constraints: mass balance + flow bounds

$$\sum_{j: ij \in A} x_{ij} - \sum_{j: ji \in A} x_{ji} = b(i) \quad \forall i \in V$$

N node arc incidence matrix

$$0 \leq x_{ij} \leq u_{ij}$$

(assumption: all values are integer, we can multiply if rational)

	x_{e_1}	x_{e_2}	$\dots$	x_{ij}	$\dots$	x_{e_m}		
	c_{e_1}	c_{e_2}	$\dots$	c_{ij}	$\dots$	c_{e_m}		
1	-1	$.$	$\dots$	$.$	$\dots$	$.$	$=$	b_1
2	$.$	$.$	$\dots$	$.$	$\dots$	$.$	$=$	b_2
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
i	1	$.$	$\dots$	-1	$\dots$	$.$	$=$	b_i
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
j	$.$	$.$	$\dots$	1	$\dots$	$.$	$=$	b_j
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
n	$.$	$.$	$\dots$	$.$	$\dots$	$.$	$=$	b_j
e_1	-1						$\geq$	$-u_1$
e_2		-1					$\geq$	$-u_2$
$\vdots$	$\vdots$	$\ddots$					$\geq$	$\vdots$
(i,j)				-1			$\geq$	$-u_{ij}$
$\vdots$	$\vdots$	$\ddots$					$\geq$	$\vdots$
e_m						-1	$\geq$	$-u_m$

Special cases

Shortest path problem path of minimum cost from s to t with costs $\lesseqgtr 0$

$b(s) = -1, b(t) = 1, b(i) = 0$

if to any other node? $b(s) = n - 1, b(i) = -1, u_{ij} = n - 1$

Max flow problem incur no cost but restricted by bounds

steady state flow from s to t

$b(i) = 0 \ \forall i \in V, \qquad c_{ij} = 0 \ \forall ij \in A \qquad ts \in A$

$c_{ts} = -1, \qquad u_{ts} = \infty$

Assignment problem min weighted bipartite matching,

$|V_1| = |V_2|, A \subseteq V_1 \times V_2$

c_{ij}

$b(i) = -1 \ \forall i \in V_1 \qquad b(i) = 1 \ \forall i \in V_2 \qquad u_{ij} = 1 \ \forall ij \in A$

Special cases

Transportation problem/Transhipment distribution of goods, warehouses-costumers

$|V_1| \neq |V_2|, \qquad u_{ij} = \infty \ \forall ij \in A$

$$\begin{aligned} \min \textstyle\sum c_{ij} x_{ij} & \\ \textstyle\sum_i x_{ij} &\geq b_j \ \forall j \\ \textstyle\sum_j x_{ij} &\leq a_i \ \forall i \\ x_{ij} &\geq 0 \end{aligned}$$

Min cost circulation problem $b(i) = 0 \ \forall i \in V$

Minimum spanning tree connected acyclic graph that spans all nodes

Multi-commodity flow problem ship several commodities using the same network, different origin destination pairs separate mass balance constraints, share capacity constraints, min overall flow

$$
\begin{aligned}
\min \textstyle\sum_k c^k x^k & & \\
N x^k \geq b^k & \qquad \forall k \\
\textstyle\sum_k x_{ij}^k \geq u_{ij} & \qquad \forall ij \in A \\
0 \leq x_{ij}^k \leq u_{ij}^k &
\end{aligned}
$$

How does the structure of the matrix looks like? Is it still TUM?

Shortest Path - Dual LP

$$
\begin{aligned}
z = \min & \sum_{ij \in A} c_{ij} x_{ij} & & \\
& \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = -1 & \textit{for } i = s & \quad (\pi_s) \\
& \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = 0 & \forall i \in V \setminus \{s, t\} & \quad (\pi_i) \\
& \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = 1 & \textit{for } i = t & \quad (\pi_t) \\
& x_{ij} \geq 0 & \forall ij \in A &
\end{aligned}
$$

Dual problem:

$$
\begin{aligned}
w^{LP} &= \max \pi_t - \pi_s & \\
& \pi_j - \pi_i \leq c_{ij} & \forall ij \in A
\end{aligned}
$$

Hence, the shortest path can be found by potential values π_i on nodes such that $\pi_s = 0, \pi_t = z$ and $\pi_j - \pi_i \leq c_{ij}$ for $ij \in A$

Maximum (s,t)-Flow

Adding a backward arc from t to s:

$$
\begin{aligned}
z = \max x_{ts} & & \\
\sum_{j:ij\in A} x_{ij} - \sum_{j:ji\in A} x_{ji} = 0 & \quad \forall i \in V & (\pi_i) \\
x_{ij} \leq u_{ij} & \quad \forall ij \in A & (w_{ij}) \\
x_{ij} \geq 0 & \quad \forall ij \in A &
\end{aligned}
$$

Dual problem:

$$w^{LP} = \min \sum_{ij\in A} u_{ij} w_{ij} \tag{1}$$

$$y_i - y_j + w_{ij} \geq 0 \quad \forall ij \in A \tag{2}$$

$$y_t - y_s \geq 1 \tag{3}$$

$$z_{ij} \geq 0 \quad \forall ij \in A \tag{4}$$

- Without (3) all potentials would go to 0.
- Keep w low because of objective function
- Keep all potentials low $\rightsquigarrow$ (3) $y_s = 1, y_t = 0$
- Cut: on left =1 on right =0. Where is the transition?
- Var w identifies the cut $\rightsquigarrow y_j - y_i + w_{ij} \geq 0 \rightsquigarrow w_{ij} = 1$

$$w_{ij} = \begin{cases} 1 & \textit{if } ij \in T \\ 0 & \textit{otherwise} \end{cases}$$

  for those arcs that minimize the cut capacity $\sum_{ij \in A} u_{ij} w_{ij}$
- Complementary slackness: $z_{ij} = 1 \implies x_{ij} = u_{ij}$

Theorem

A strong dual to the max (st)-flow is the minimum (st)-cut problem:

$$\min_X \{ \sum_{ij \in A: i \in X, j \notin X} u_{ij} : s \in X \subset V \setminus \{t\} \}$$

Min Cost Flow - Dual LP

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b_i \qquad \forall i \in V \qquad (y_i)$$

$$x_{ij} \le u_{ij} \qquad \forall ij \in A \qquad (z_{ij})$$

$$x_{ij} \ge 0 \qquad \forall ij \in A$$

Dual problem:

$$\max \sum_{i \in V} b_i y_i - \sum_{ij \in E} u_{ij} z_{ij} \tag{5}$$

$$-c_{ij} - y_i + y_j \le z_{ij} \forall ij \in E \qquad (x_{ij}) \tag{6}$$

$$z_{ij} \ge 0 \forall ij \in A \tag{7}$$

- reduced costs $\bar{c}_{ij} = c_{ij} + y_j - y_i$, hence (2) $-\bar{c}_{ij} \leq z_{ij}$
- $u_e = \infty$ then $z_e = 0$ (from obj. func) and $\bar{c}_{ij} \geq 0$ (optimality condition)
- $u_e < \infty$ then $z_e \geq 0$ and $z_e \geq -\bar{c}_{ij}$ then $z_e = \max\{0, -\bar{c}_{ij}\}$, hence z_e is determined by others and may be skipped
- Complementary slackness
  (at optimality: each primal variable $\times$ the corresponding dual slack must be equal 0, ie, $x_e(\bar{c}_e + z_e) = 0$; each dual variable $\times$ the corresponding primal slack must be equal 0, ie, $z_e(x_e - u_e) = 0$)
  - $x_e > 0$ then $-\bar{c}_e = z_e$ then $\max\{0, \bar{c}_e\}$ then $-\bar{c}_e > 0$ then ($\bar{c}_e < 0$ then $x_c > 0$)
  - $z_e > 0$ then $x_e = u_e$ then $(-\bar{c} > 0$ then $x_e = u_e)$

  Hence:

  $\bar{c}_e < 0$ then $x_e = u_e \neq \infty$

  $\bar{c}_e > 0$ then $x_e = 0$

Summary

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