DM811

Heuristics for Combinatorial Optimization

Lecture 15
Methods for Experimental Analysis

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Course Overview

- ✔ Combinatorial Optimization, Methods and Models
- ✔ CH and LS: overview
- ✔ Working Environment and Solver Systems
- ~ Methods for the Analysis of Experimental Results
- ✔ Construction Heuristics
- ✔ Local Search: Components, Basic Algorithms
- ✔ Local Search: Neighborhoods and Search Landscape
- ✔ Efficient Local Search: Incremental Updates and Neighborhood Pruning
- ✔ Stochastic Local Search & Metaheuristics
- ~ Configuration Tools: F-race
- Very Large Scale Neighborhoods

Examples: GCP, CSP, TSP, SAT, MaxIndSet, SMTWP, Steiner Tree, Unrelated Parallel Machines, p-median, set covering, QAP, ...

Outline

1. Experimental Methods: Inferential Statistics
    - Statistical Tests
    - Experimental Designs
    - Applications to Our Scenarios

2. Race: Sequential Testing

Outline

Inferential Statistics

- We work with samples (instances, solution quality)
- But we want sound conclusions: generalization over a given population (all runs, all possible instances)
- Thus we need statistical inference

Random Sample	Inference	Population
X^n	$\longrightarrow$	$P(x, \theta)$
Statistical Estimator $\widehat{\theta}$		Parameter θ

Since the analysis is based on finite-sized sampled data, statements like

> *"the cost of solutions returned by algorithm $\mathcal{A}$ is smaller than that of algorithm $\mathcal{B}$"*

must be completed by

> *"at a level of significance of 5%".*

A Motivating Example

- There is a competition and two stochastic algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$ are submitted.
- We run both algorithms once on n instances.
  On each instance either $\mathcal{A}_1$ wins (+) or $\mathcal{A}_2$ wins (-) or they make a tie (=).

Questions:

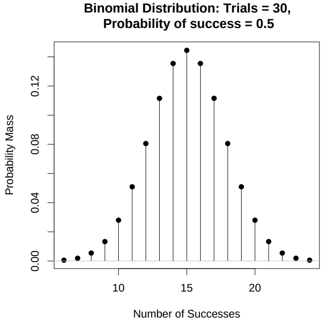
1. If we have only 10 instances and algorithm $\mathcal{A}_1$ wins 7 times how confident are we in claiming that algorithm $\mathcal{A}_1$ is the best?
2. How many instances and how many wins should we observe to gain a confidence of 95% that the algorithm $\mathcal{A}_1$ is the best?

A Motivating Example

- p: probability that $\mathcal{A}_1$ wins on each instance (+)
- n: number of runs without ties
- Y: number of wins of algorithm $\mathcal{A}_1$

If each run is indepenedent and consitent:

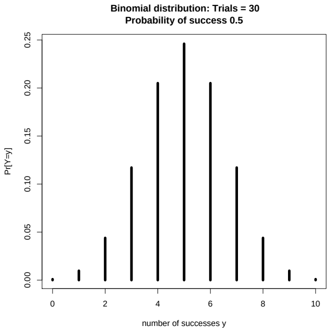
$$Y \sim B(n,p): \qquad \Pr[Y=y] = \binom{n}{y} p^y (1-p)^{n-y}$$



1 If we have only 10 instances and algorithm $\mathcal{A}_1$ wins 7 times how confident are we in claiming that algorithm $\mathcal{A}_1$ is the best?

Under these conditions, we can check how unlikely the situation is if it were $p(+) \leq p(-)$.

If $p = 0.5$ then the chance that algorithm $\mathcal{A}_1$ wins 7 or more times out of 10 is 17.2%: quite high!



2 How many instances and how many wins should we observe to gain a confidence of 95% that the algorithm $\mathcal{A}_1$ is the best?

To answer this question, we compute the 95% quantile, *i.e.*, $y : \Pr[Y \geq y] < 0.05$ with $p = 0.5$ at different values of n:

n	10	11	12	13	14	15	16	17	18	19	20
y	9	9	10	10	11	12	12	13	13	14	15

This is an application example of sign test, a special case of binomial test in which $p = 0.5$

Statistical tests

General procedure:

- Assume that data are consistent with a null hypothesis H_0 (e.g., sample data are drawn from distributions with the same mean value).
- Use a statistical test to compute how likely this is to be true, given the data collected. This "likely" is quantified as the p-value.
- Do not reject H_0 if the p-value is larger than an user defined threshold called level of significance α.
- Alternatively, (p-value $< \alpha$), H_0 is rejected in favor of an alternative hypothesis, H_1, at a level of significance of α.

Inferential Statistics

Two kinds of errors may be committed when testing hypothesis:

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true})$$

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \mid H_0 \text{ is false})$$

General rule:

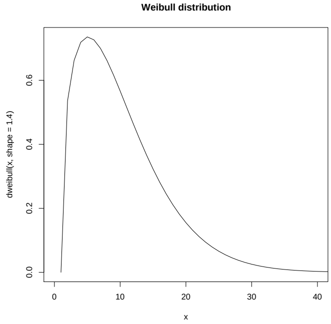
1. specify the type I error or level of significance α
2. seek the test with a suitable large statistical power, i.e., $1 - \beta = P(\text{reject } H_0 \mid H_0 \text{ is false})$

Theorem: Central Limit Theorem

If X^n is a random sample from an **arbitrary** distribution with mean μ and variance σ then the average $\bar{X}^n$ is asymptotically normally distributed, *i.e.*,

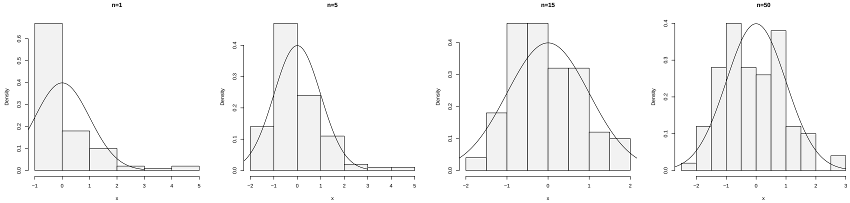
$$\bar{X}^n \approx N(\mu, \frac{\sigma^2}{n}) \qquad \text{or} \qquad z = \frac{\bar{X}^n - \mu}{\sigma/\sqrt{n}} \approx N(0,1)$$

- Consequences:
  - allows inference from a sample
  - allows to model errors in measurements: $X = \mu + \epsilon$
- Issues:
  - n should be *enough* large
  - μ and σ must be known



$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

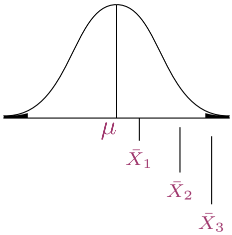
Samples of size 1, 5, 15, 50 repeated 100 times



Hypothesis Testing and Confidence Intervals

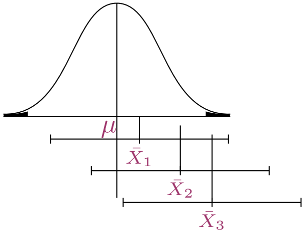
A test of hypothesis determines how likely a sampled estimate $\hat{\theta}$ is to occur under some assumptions on the parameter θ of the population.

$$Pr\left\{\mu - z_1 \frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + z_2 \frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha$$



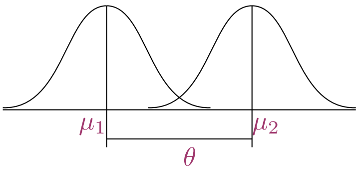
A confidence interval contains all those values that a parameter θ is likely to assume with probability $1 - \alpha$: $Pr(\hat{\theta}_1 < \theta < \hat{\theta}_2) = 1 - \alpha$

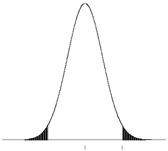
$$Pr\left\{\bar{X} - z_1 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_2 \frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha$$



Statistical Tests

The Procedure of Test of Hypothesis





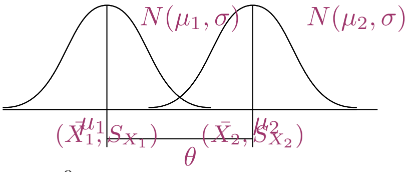
1. Specify the parameter θ and the test hypothesis,

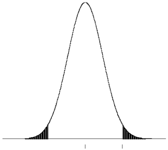
$$\theta = \mu_1 - \mu_2 \qquad \begin{cases} H_0 : \theta = 0 \\ H_1 : \theta \neq 0 \end{cases}$$

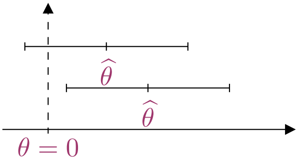
2. Obtain $P(\theta|\theta = 0)$, the null distribution of θ
3. Compare $\hat{\theta}$ with the $\alpha/2$-quantiles (for two-sided tests) of $P(\theta|\theta = 0)$ and reject or not H_0 according to whether $\hat{\theta}$ is larger or smaller than this value.

Statistical Tests

The Confidence Intervals Procedure







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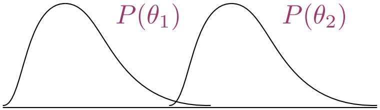
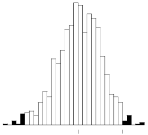
3. Determine $(\hat{\theta}^-, \hat{\theta}^+)$ such that $Pr\{\hat{\theta}^- \leq \theta \leq \hat{\theta}^+\} = 1 - \alpha$.
4. Do not reject H_0 if $\theta = 0$ falls inside the interval $(\hat{\theta}^-, \hat{\theta}^+)$. Otherwise reject H_0.

Statistical Tests
The Confidence Intervals Procedure

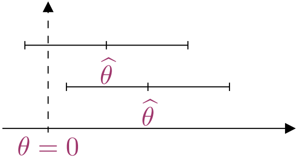




$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_{X_1} - S_{X_2}}{r}}}$

$T \sim$ Student's *t* Distribution

$\theta^* = \bar{X}_1^* - \bar{X}_2^*$



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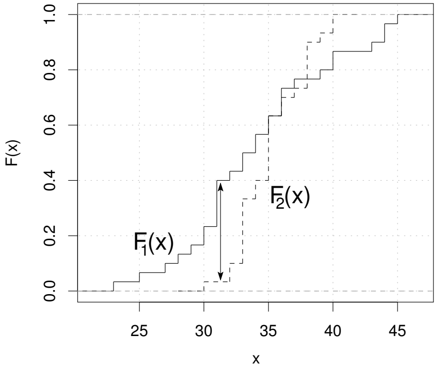
Kolmogorov-Smirnov Tests

The test compares empirical cumulative distribution functions.



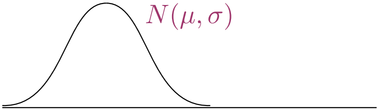
It uses maximal difference between the two curves, $sup_x|F_1(x) - F_2(x)|$, and assesses how likely this value is under the null hypothesis that the two curves come from the same data

The test can be used as a two-samples or single-sample test (in this case to test against theoretical distributions: goodness of fit)

Parametric *vs* Nonparametric

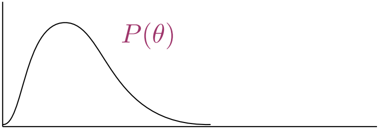
Parametric assumptions:

- independence
- homoschedasticity
- normality



Nonparametric assumptions:

- independence
- homoschedasticity



- Rank based tests
- Permutation tests
  - Exact
  - Conditional Monte Carlo

Preparation of the Experiments

Variance reduction techniques

- Blocking on instances
- Same pseudo random seed

Sample Sizes

- If the sample size is large enough (infinity) any difference in the means of the factors, no matter how small, will be significant
- Real *vs* Statistical significance
  Study factors until the improvement in the response variable is deemed small
- Desired statistical power + practical precision $\Rightarrow$ sample size

Note: If resources available for N runs then the optimal design is one run on N instances [Birattari, 2004]

The Design of Experiments for Algorithms

- Statement of the objectives of the experiment
  - Comparison of different algorithms
  - Impact of algorithm components
  - How instance features affect the algorithms
- Identification of the sources of variance
  - Treatment factors (qualitative and quantitative)
  - Controllable nuisance factors $\Leftarrow$ blocking
  - Uncontrollable nuisance factors $\Leftarrow$ measuring
- Definition of factor combinations to test

  Easiest design: Unreplicated or Replicated Full Factorial Design
- Running a pilot experiment and refine the design
  - Bugs and no external biases
  - Ceiling or floor effects
  - Rescaling levels of quantitative factors
  - Detect the number of experiments needed to obtained the desired power.

Experimental Design

Algorithms $\Rightarrow$ Treatment Factor; Instances $\Rightarrow$ Blocking/Random Factor

Design A: One run on various instances (Unreplicated Factorial)

	Algorithm 1	Algorithm 2	...	Algorithm k
Instance 1	X_{11}	X_{12}		X_{1k}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Instance b	X_{b1}	X_{b2}		X_{bk}

Design B: Several runs on various instances (Replicated Factorial)

	Algorithm 1	Algorithm 2	...	Algorithm k
Instance 1	$X_{111}, \ldots, X_{11r}$	$X_{121}, \ldots, X_{12r}$		$X_{1k1}, \ldots, X_{1kr}$
Instance 2	$X_{211}, \ldots, X_{21r}$	$X_{221}, \ldots, X_{22r}$		$X_{2k1}, \ldots, X_{2kr}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Instance b	$X_{b11}, \ldots, X_{b1r}$	$X_{b21}, \ldots, X_{b2r}$		$X_{bk1}, \ldots, X_{bkr}$

Multiple Comparisons

$H_0: \ \mu_1 = \mu_2 = \mu_3 = \ldots$ $\quad$ $H_1:$ {at least one differs}

Applying a statistical test to all pairs the error of Type I is not α but higher:

$$\alpha_{EX} = 1 - (1 - \alpha)^c$$

Eg, for $\alpha = 0.05$ and $c = 3 \Rightarrow \alpha_{EX} = 0.14$!

Adjustment methods

- Protected versions: global test + no adjustments
- Bonferroni $\alpha = \alpha_{EX}/c$ (conservative)
- Tukey Honest Significance Method (for parametric analysis)
- Holm (step-wise)
- Other step procedures

Post-hoc analysis: Once the effect of factors has been recognized a finer grained analysis is performed to distinguish where important differences are.

Statistical Tests
Univariate Analysis

Several runs on a single instance

Global tests	Replicated
Parametric	F-test
Non-Parametric Rank based	Kruskall-Wallis Test
Non-Parametric Permutation based	Pooled Permutations
Non-Parametric KS type	Birnbaum-Hall test

Statistical Tests
Univariate Analysis

Several runs on a single instance

Pairwise tests	Replicated
Parametric	t-test Tukey HSD
Non-Parametric Rank based	Kruskall-Wallis Test or Mann-Whitney test ≡ *Wilcoxon Rank Sum Test* or Binomial test
Non-Parametric Permutation based	Pooled Permutations
Non-Parametric KS type	Birnbaum-Hall test

- Matched pairs versions: when, when not
- t-test with different variances

On various instances (Designs A and B)

Global tests	Unreplicated (Design A)	Replicated (Design B)
Parametric	F-test	F-test
Non-Parametric Rank based	Friedman Test	Friedman Test
Non-Parametric Permutation based	Simple Permutations	Synchronized Permutations

Statistical Tests
Univariate Analysis

On various instances (Designs A and B)

Pairwise tests	Unreplicated	Replicated
Parametric	t-test Tukey HSD	t-test Tukey HSD
Non-Parametric Rank based	Friedman Test or *Wilcoxon Signed Rank Test*	Friedman Test
Non-Parametric Permutation based	Simple Permutations	Synchronized Permutations

- Matched pairs versions: when, when not
- t-test Welch variant: no assumption of equal variances

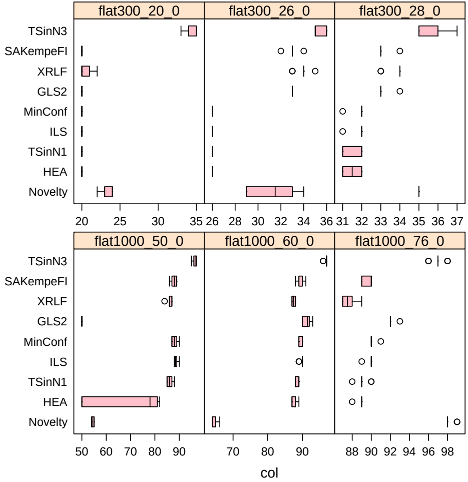
An Example

SLS algorithms for Graph Coloring:
Results collected on a set of benchmark instances

Instance	HEA		TS_{N1}		ILS		MinConf		XRLF	
Instance	Succ.	k	Succ.	k	Succ.	k	Succ.	k	Succ.	k
flat300_20_0	10	20	10	20	10	20	10	20	6	20
flat300_26_0	10	26	10	26	10	26	10	26	1	33
flat300_28_0	6	31	4	31	2	31	1	31	1	34
flat1000_50_0	4	50	2	85	6	88	4	87	1	84
flat1000_60_0	4	87	3	88	1	89	4	89	6	87
flat1000_76_0	1	88	1	88	1	89	8	90	6	87
	GLS		SA_{N2}		Novelty		TS_{N3}			
Instance	Succ.	k	Succ.	k	Succ.	k	Succ.	k		
flat300_20_0	10	20	10	20	1	22	1	33		
flat300_26_0	10	33	1	32	4	29	6	35		
flat300_28_0	8	33	8	33	10	35	4	35		
flat1000_50_0	10	50	1	86	6	54	1	95		
flat1000_60_0	4	90	1	88	4	64	1	96		
flat1000_76_0	8	92	4	89	8	98	1	96		

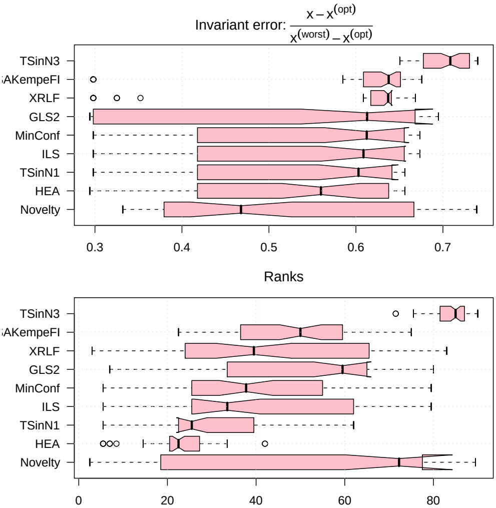
An Example

Raw data on the instances:



```
> load("gcp-all-classes.dataR")
> G <- F[F$class=="Flat",]
> bwplot(alg ~ col | inst,data=G,scales=list(x=list(relation="free")),pch="|")
> boxplot(err3~alg,data=G,horizontal=TRUE,main=expression(paste("Invariant error: ",frac(x-x
    ^(opt),x^(worst)-x^(opt)))),notch=TRUE,col="pink")
> boxplot(rank~alg,data=G,horizontal=TRUE,main="Ranks",notch=TRUE,col="pink")
```

An Example

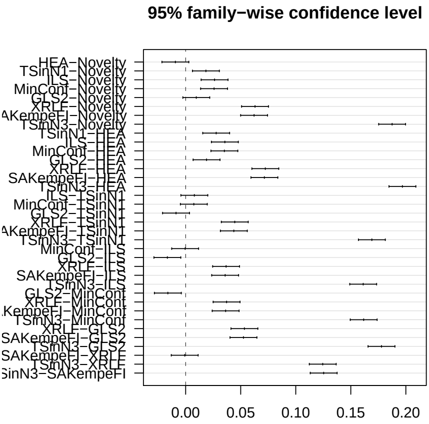


Note: notches are not appropriate for comparative inference

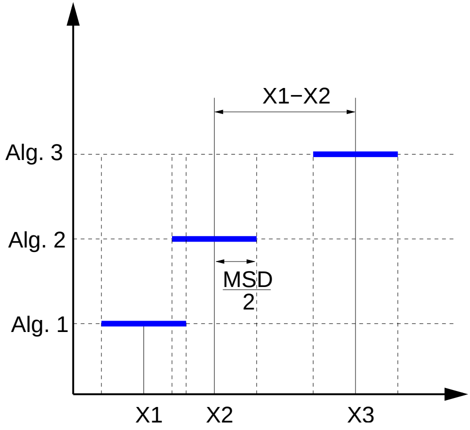
```
> pairwise.wilcox.test(G$err3,G$alg,paired=TRUE)

Pairwise comparisons using Wilcoxon rank sum test

data: G$err3 and G$alg
```

```
          Novelty HEA     TSinN1  ILS     MinConf GLS2    XRLF    SAKempeFI
HEA       1.00000 -       -       -       -       -       -       -
TSinN1    1.00000 0.00413 -       -       -       -       -       -
ILS       1.00000 1.3e-05 0.00072 -       -       -       -       -
MinConf   1.00000 9.4e-06 0.00042 1.00000 -       -       -       -
GLS2      1.00000 0.11462 0.94136 1.00000 1.00000 -       -       -
XRLF      0.25509 1.7e-05 0.02624 0.72455 0.47729 1.00000 -       -
SAKempeFI 0.72455 1.4e-07 3.0e-06 0.02708 0.02113 1.00000 1.00000 -
TSinN3    3.7e-08 5.8e-10 5.8e-10 5.8e-10 5.8e-10 5.8e-10 5.8e-10 5.8e-10

P value adjustment method: holm
```

```
> par(las=1,mar=c(3,8,3,1))
> plot(TukeyHSD(aov(err3~alg*inst,data=G),which="alg"),las=1,mar=c(3,7,3,1))
```



An Example

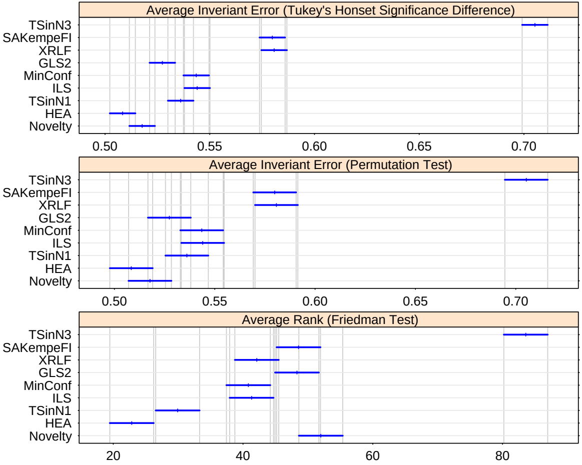


Minimal Significant Difference (MSD)

interval that satisfies simultaneously each comparison

Differences are statistically significant if the confidence intervals do not overlap

An Example


Average Inveriant Error (Tukey's Honset Significance Difference)
TSinN3
SAKempeFI
XRLF
GLS2
MinConf
ILS
TSinN1
HEA
Novelty
0.50
0.55
0.60
0.65
0.70
Average Inveriant Error (Permutation Test)
TSinN3
SAKempeFI
XRLF
GLS2
MinConf
ILS
TSinN1
HEA
Novelty
0.50
0.55
0.60
0.65
0.70
Average Rank (Friedman Test)
TSinN3
SAKempeFI
XRLF
GLS2
MinConf
ILS
TSinN1
HEA
Novelty
20
40
60
80

Outline

Unreplicated Designs

Procedure Race [Birattari 2002]:
repeat
    Randomly select an unseen instance and run all candidates on it
    Perform *all-pairwise comparison* statistical tests
    Drop all candidates that are significantly inferior to the best algorithm
until only one candidate left or no more unseen instances ;

- F-Race use Friedman test
- Holm adjustment method is typically the most powerful

```
race(wrapper.file, maxExp=0,
            stat.test=c("friedman","t.bonferroni","t.holm","t.none"),
               conf.level=0.95, first.test=5, interactive=TRUE,
                  log.file="", no.slaves=0,...)
```

Sequential Testing