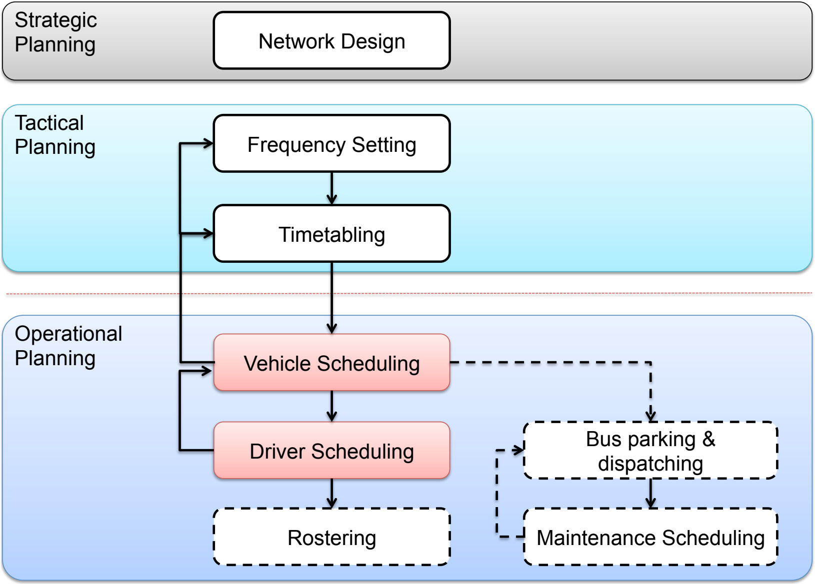
Crew Scheduling: Models and Algorithms

Stefano Gualandi
Università di Pavia, Dipartimento di Matematica

email: stefano.gualandi@unipv.it
twitter: @famo2spaghi
blog: http://stegua.github.com

Overview of Planning Activities (Desaulniers&Hickman2007)



Crew Scheduling

Definition (Relief times)

Each **vehicle duty** (herein called **block**) has a set of **relief times** where a driver substitution may occur.

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Crew Scheduling

Definition (Piece of Work (PoW))

A piece of work p is a continuous driving period from $s(p)$ to $e(p)$. A piece of work is feasible for a block k if both $s(p)$ and $e(p)$ are relief times of k.

Example: Given

- a block that starts at $8:30$ and ends $12:30$
- relief times at $\{8:30, 9:30, 10:20, 11:20, 12:30\}$
- constraint: a PoW last at least 01:00 and at most 02:00

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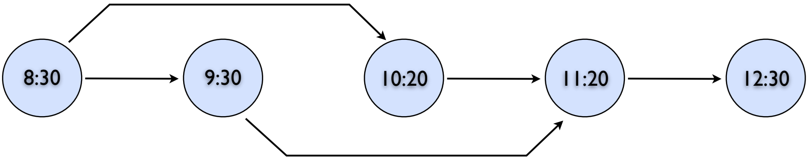

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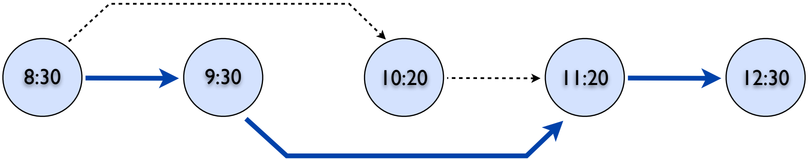

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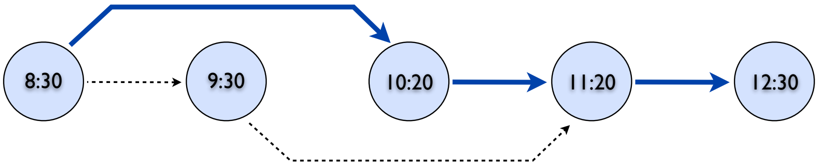


Crew Scheduling

Definition (Crew duty)

A crew duty consists of a set of pairs (p, k) where p is a piece of work associated to block k.

Definition (Crew Scheduling)

Given a Vehicle Schedule (i.e. a collection of vehicle duties), the **Crew Scheduling** problem consists of finding a set of crew duties to be assigned to drivers in order to guarantee the daily service.

Crew Scheduling: Urban and Regional

Crew Scheduling

- $T_k = \{t_1^k, \ldots, t_{u_k}^k\}$ is the set of relief times for block k
- t_1^k and $t_{u_k}^k$ are the starting and ending time of the block k
- P_k set of piece of work feasible for block k
- $\mathcal{D} = \{d_1, \ldots, d_{|\mathcal{D}|}\}$ set of all feasible duties

Partition of blocks into piece of works

For each block, we define the network $G_k = (N_k, A_k)$ where

- $N_k = T_k$ one node for each relief time
- $A_k = \{(s(p), e(p)) \mid p \in P_k\}$ an arc for each piece of work

The problem of finding a partition of a block into piece of works is:

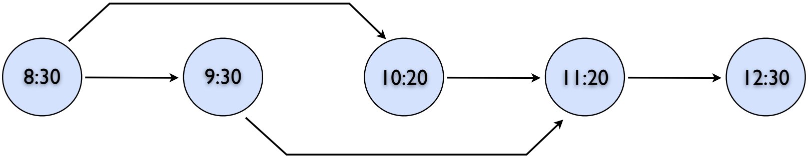
$$-\sum_{p \in P_K | e(p)=i} y_p^k + \sum_{p \in P_k | s(p)=i} y_p^k = \begin{cases} 1 & \text{if } i = t_1^k \\ 0 & \text{if } i = t_j^k, j = 2, \ldots, u_k - 1 \\ -1 & \text{if } i = t_{u_k}^k \end{cases}$$

$$y_p^k \in \{0, 1\} \qquad \forall p \in P_k$$

We can write in compact form:

$$E^k y^k = b^k, \qquad y^k \in \{0, 1\}$$

Partition of blocks into piece of works



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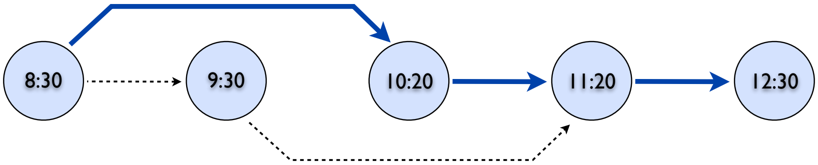

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Crew Scheduling: Basic Model

- Let x be a $|\mathcal{D}|$-vector of binary variables corresponding to the set of all feasible duties
- Let I_{pk} be the subset of all the duty indices corresponding in G to arcs incident to (p, k)

$$
\begin{aligned}
\min \quad & \sum_{d \in \mathcal{D}} c_d x_d && && (1)\\
\text{s.t.} \quad & E^k y^k = b^k && && (2)\\
& \sum_{d \in I_{pk}} x_d = y_p^k && \forall p \in P_k, k = 1, \ldots, r && (3)\\
& y^k \in \{0,1\}^{m_k} && \forall k = 1, \ldots, r && (4)\\
& x \in \{0,1\}^{|\mathcal{D}|} && && (5)\\
& x \in X. && && (6)
\end{aligned}
$$

Crew Scheduling and Regional Transit

In Regional Transit, Crew Scheduling is performed before of Vehicle Scheduling, and in practice the set of pieces of work is given (there are very few relief times).

- Let P be the set of piece of work
- Let $\mathcal{D}$ be the set of every possible duty
- The cost of a duty j is denoted by c_j
- $b_{ij} = \begin{cases} 1 & \text{if the piece of work } i \text{ appears in duty } j \\ 0 & \text{otherwise} \end{cases}$

Crew Scheduling and Regional Transit

$$\min \sum_{j \in \mathcal{D}} c_j \lambda_j \tag{7}$$

$$\text{s.t.} \sum_{j \in D} b_{ij} \lambda_j = 1 \qquad \forall i \in P \quad \rightarrow \quad \text{partition of PoW} \tag{8}$$

$$\lambda_j \in \{0, 1\} \qquad \forall j \in \mathcal{D} \quad \rightarrow \quad \text{every possible duty} \tag{9}$$

"The set partitioning problem is arguably the easiest optimization model in the world to represent on paper"

"In contrast, the real-life computer code used to manage this simple model can easily run in the order of many hundred thousand lines"

Crew Scheduling: Set Partitioning Formulation

$$\min \sum_{j \in D} c_j \lambda_j \tag{10}$$

$$\text{s.t.} \sum_{j \in D} b_{ij} \lambda_j = 1 \quad \forall i \in P \quad \rightarrow \quad \text{partition of PoW} \tag{11}$$

$$\lambda_j \geq 0 \quad \forall j \in \mathcal{D} \quad \rightarrow \quad \text{every possible duty} \tag{12}$$

First step: to solve the continuous relaxation

QUESTION: Is it easy to solve the LP?

ISSUE: the size of $\mathcal{D}$ is exponential in $|P|$!

Column Generation

$$(LP) \quad \min \{cx \mid Ax \geq b, x \in \mathbb{R}^n\}$$

- Column Generation is an efficient algorithm for solving **very large linear programs as (LP-MP)**
- Since most of the variables will be **non-basic** and assume a value of zero in the optimal solution, only a subset of variables need to be considered
- Column generation leverages this idea to generate only the variables which have the potential to improve the objective function, that is, to find variables with negative reduced cost

Dealing with Finitely Many Columns

The main idea is to start with a subset of columns $\bar{\mathcal{D}} \subset \mathcal{D}$ such that a feasible solution to the following problem exists:

$$z_{RMP} = \min \sum_{j \in \bar{\mathcal{D}}} c_j \lambda_j \quad (13)$$

$$\text{s.t.} \sum_{j \in \bar{\mathcal{D}}} b_{ij} \lambda_j \geq 1 \qquad \forall i \in P \quad (14)$$

$$\lambda_j \geq 0 \qquad \forall j \in \bar{\mathcal{D}} \quad (15)$$

Using the Duality Theory of Linear Programming with can generate as set of improving columns. . .

Column Generation: A Dual Persepective

Consider the LP relaxation of the "master" problem and its dual:

$$(P)\ \min \sum_{j\in\bar{\mathcal{D}}} c_j\lambda_j \quad \text{s.t.} \sum_{j\in\bar{\mathcal{D}}} b_{ij}\lambda_j \geq 1, \quad \forall i \in P, \qquad \lambda_j \geq 0, \quad \forall j \in \bar{\mathcal{D}}.$$

$$(D)\ \max \sum_{i\in P} \pi_i \quad \text{s.t.} \sum_{i\in P} b_{ij}\pi_i \leq c_j, \quad \forall j \in \bar{\mathcal{D}}, \qquad \pi_i \geq 0, \quad \forall i \in P.$$

Using the Duality Theory of Linear Programming with can generate as set of improving columns... by separating inequalities on the dual of the master problem!

Pricing Subproblem (Separation on the Master Dual)

The question is:

Does a column (duty) in $\mathcal{D} \setminus \bar{\mathcal{D}}$ that could improve the current optimal solution of the linear relaxation exist?

Does a column (row of the dual) exist such that ...?

$$\exists j \in \mathcal{D} \setminus \bar{\mathcal{D}} : \quad \sum_{i \in P} b_{ij} \pi_i > c_j$$

Pricing Subproblem (Separation on the Master Dual)

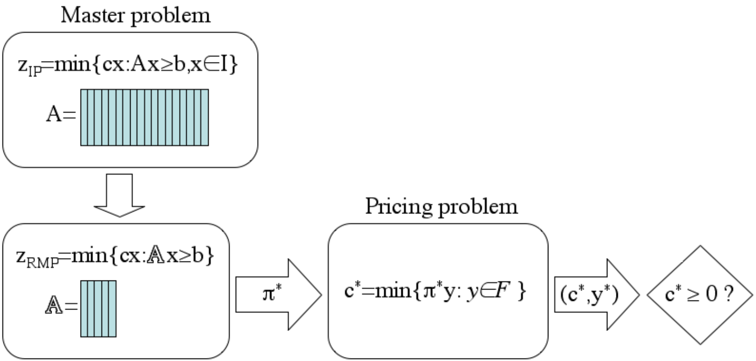
Given the vector of optimal dual multipliers $\bar{\pi}$ for (RMP), we look for a column (duty) such that:

$$\begin{aligned} c^* = \min \quad & c_j - \sum_{i \in P} \bar{\pi}_i y_i \\ \text{s.t.} \quad & y \in F \\ & y_i \in \{0, 1\}. \end{aligned}$$

If $c^* < 0$, the vector of variables y is the incidence vector of an *"improving"* column. It corresponds to a variable with **negative reduced cost** in the (restricted) master problem.

What is F in Crew Scheduling problems?

Column Generation: Algorithmic Persepective



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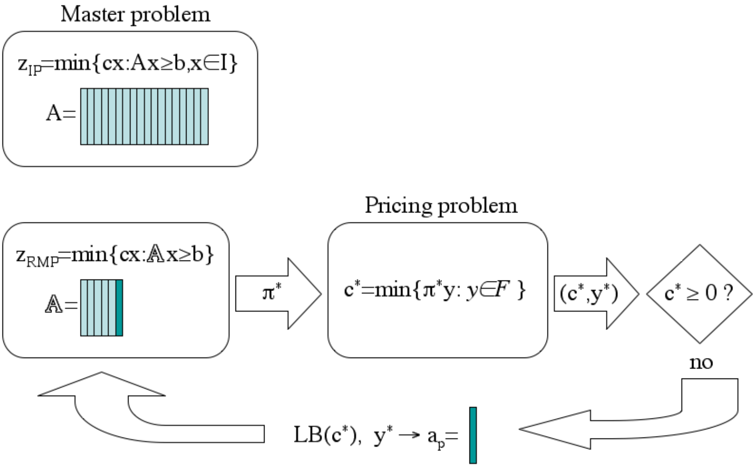

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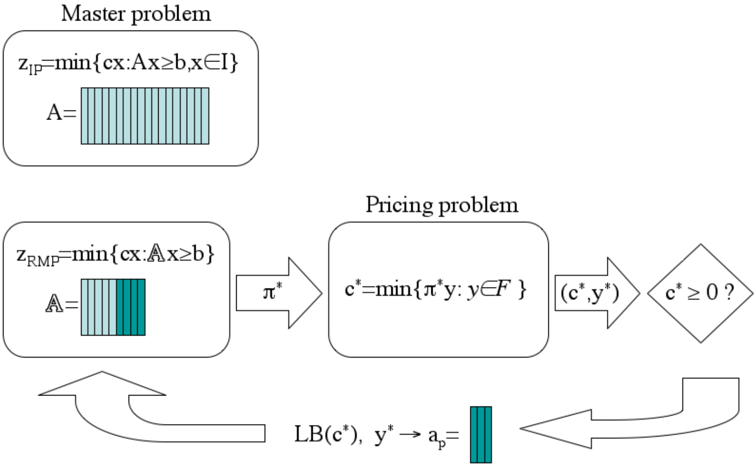

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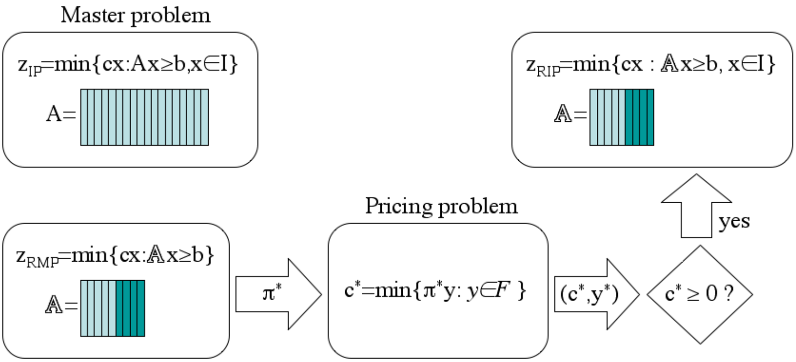

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