DM545

Linear and Integer Programming

Lecture 1

Introduction

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Outline

Context

Students:

- Computer Science (3rd year)
- Applied Mathematics (3rd year)
- Math-economy (3rd year)

Prerequisites

- Calculus (MM501, MM502)
- Linear Algebra (MM505)

Practical Information

Teacher: Marco Chiarandini (marco@imada.sdu.dk)
Instructor: Christian Nørskov (cnoer10@student.sdu.dk)

Schedule ($\approx$ 24 lecture hours + $\approx$ 20 exercise hours):

Week	15	16	17	18	19	20	21	22	23
Tir, 12-14	Intro (Fælles) (U150)			Intro (Fælles) (U150)	Intro (Fælles) (U150)	Intro (Fælles) (U150)	Intro (Fælles) (U150)	Intro (Fælles) (U150)	
Tir, 14-16								Træning (S1) (U20)	
Tir, 15-17			Intro (Fælles) (U47)						
Ons, 10-12						Intro (Fælles) (U91)			
Ons, 12-14	Intro (Fælles) (U71)		Lab (S1) (Terminalrum)	Træning (S1) (U56)	Intro (Fælles) (U20)	Træning (S1) (U20)	Træning (S1) (U20)	Træning (S1) (U20)	
Fre, 10-12	Træning (S1) (U42)		Intro (Fælles) (U42)		Træning (S1) (U42)		Intro (Fælles) (U42)	Intro (Fælles) (U42)	Træning (S1) (U42)

Practical Information

Communication tools

- BlackBoard (BB) $\Leftrightarrow$ Public Web Page (WWW)
  (link from http://www.imada.sdu.dk/~marco/DM545/)
- Announcements in BlackBoard
- Classes
- Personal email

Text book

HL Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

Other references:

MG J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

Wo L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

FGK Robert Fourer, David M. Gay, and Brian W. Kernighan, AMPL: A Modeling Language for Mathematical Programming. Duxbury Press, Brooks Cole, Publishing Company, 2003.

Public Web Page is the main reference for list of contents (pensum).
It Contains:

- slides (text missing)
- list of topics
- links
- software

Contents

Evaluation

- 5 ECTS

- obligatory Assignments, pass/fail, evaluation by teacher (2 hand ins)
  practical part
  modeling + programming in AMPL

- 4 hour written exam, 7-grade scale, external censor
  (theory part)
  similar to exercises in class
  on June 10

- (language: English and Danish)

Obligatory Assignments

- Small projects (in groups of 2) must be passed to attend the written exam
- They require the use of the AMPL system + CPLEX or Gurobi
  Software available for all systems from the Public Web Page:
  ''Software and Data'' $\rightarrow$ ''AMPL''
  (Get the password in class)

Training Sessions

- Prepare them in advance to get out the most
- Best carried out in small groups
- Exam rehearsal (first days of June)

Outline

What is OR?

Operation Research (aka, Management Science, Analytics):
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system,
usually under conditions requiring the allocation of scarce resources,
by means of **mathematics** and **computer science**.

Quantitative methods for planning and analysis.

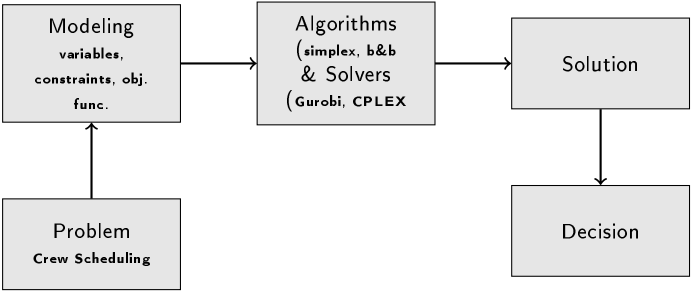
Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
  - Fewest number of people
  - Shortest route
- Not all plans are feasible - there are constraining rules
  - Limited amount of available resources
- It can be extremely difficult to figure out what to do

OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Central Idea

Build a mathematical model describing exactly what one wants, and what the "rules of the game" are. However, **what is a mathematical model and how?**

Mathematical Modeling

- Find out exactly what the decision makes needs to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Decision Variables of suitable type (continuous, integer valued, binary) corresponding to the needs
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

Outline

Resource Allocation

In manufacturing industry, factory planning: find the best product mix.

Example

A factory makes two products standard and deluxe.

A unit of standard gives a profit of 6k Dkk.
A unit of deluxe gives a profit of 8k Dkk.

The grinding and polishing times in terms of hours per week for a unit of each type of product are given below:

	Standard	Deluxe
(Machine 1) Grinding	5	10
(Machine 2) Polishing	4	4

Grinding capacity: 60 hours per week
Polishing capacity: 40 hours per week
Q: How much of each product, standard and deluxe, should we produce to maximize the profit?

Mathematical Model

Decision Variables

$x_1 \geq 0$ units of product standard
$x_2 \geq 0$ units of product deluxe

Object Function

$\max\ 6x_1 + 8x_2$ maximize profit

Constraints

$$
\begin{array}{rcrcrl}
5x_1 & + & 10x_2 & \leq & 60 & \text{Grinding capacity} \\
4x_1 & + & 4x_2 & \leq & 40 & \text{Polishing capacity}
\end{array}
$$

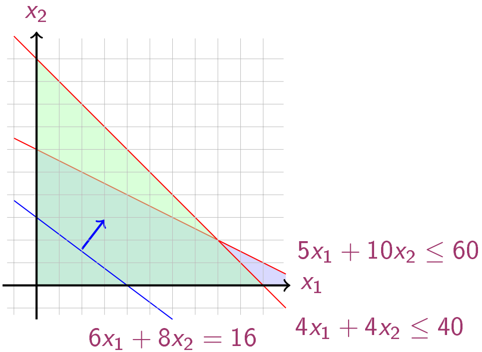
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$
\begin{array}{rrcrcr}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$1, 2, \ldots, n$	products
$1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
$c_j = \sigma_j - \sum_{i=1}^{n} \rho_i a_{ij}$	profit per unit of product j
ρ_i	prevailing market value for material i
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1 x_1 & + & c_2 x_2 & + & c_3 x_3 & + & \ldots & + & c_n x_n & = & z \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + & a_{13} x_3 & + & \ldots & + & a_{1n} x_n & \leq & b_1 \\
& a_{21} x_1 & + & a_{22} x_2 & + & a_{23} x_3 & + & \ldots & + & a_{2n} x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1} x_1 & + & a_{m2} x_2 & + & a_{m3} x_3 & + & \ldots & + & a_{mn} x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1 x_1 & + & c_2 x_2 & + & c_3 x_3 & + & \ldots & + & c_n x_n & = & z \\
\text{s.t.} & a_{11} x_1 & + & a_{12} x_2 & + & a_{13} x_3 & + & \ldots & + & a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + & a_{23} x_3 & + & \ldots & + & a_{2n} x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + & a_{m3} x_3 & + & \ldots & + & a_{mn} x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
c^T = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & c^T x \\
 & Ax & \leq & b \\
 & x & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \ldots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max \quad 6x_1 &+ 8x_2 \\ 5x_1 &+ 10x_2 \leq 60 \\ 4x_1 &+ 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

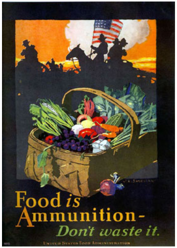
$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
    eat enough but not too much of Vitamin A
    eat enough but not too much of Sodium
    eat enough but not too much of Calories
    ...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread, and
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50,000)

Food	Cost per serving	Vitamin A	Calories
Corn	\$0.18	107	72
2% Milk	\$0.23	500	121
Wheat Bread	\$0.05	0	65

The Mathematical Model

Parameters (given data)

F = set of foods
N = set of nutrients

a_{ij} = amount of nutrient j in food i, $\forall i \in F$, $\forall j \in N$
c_i = cost per serving of food $i, \forall i \in F$
F_{mini} = minimum number of required servings of food $i, \forall i \in F$
F_{maxi} = maximum allowable number of servings of food $i, \forall i \in F$
N_{minj} = minimum required level of nutrient $j, \forall j \in N$
N_{maxj} = maximum allowable level of nutrient $j, \forall j \in N$

Decision Variables

x_i = number of servings of food i to purchase/consume, $\forall i \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{i \in F} c_i x_i$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \forall j \in N$$

Constraint Set 2: For each nutrient $j \in N$, do not exceed the maximum allowable level.

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \forall j \in N$$

Constraint Set 3: For each food $i \in F$, select at least the minimum required number of servings

$$x_i \geq F_{mini}, \forall i \in F$$

Constraint Set 4: For each food $i \in F$, do not exceed the maximum allowable number of servings.

$$x_i \leq F_{maxi}, \forall i \in F$$

The Mathematical Model

system of equalities and inequalities

$$\min \quad \sum_{i \in F} c_i x_i$$

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \qquad \forall j \in N$$

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \qquad \forall j \in N$$

$$x_i \geq F_{mini}, \qquad \forall i \in F$$

$$x_i \leq F_{maxi}, \qquad \forall i \in F$$

- The linear programming model consisted of 9 equations in 77 variables
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution
- The original instance:
  http://www.gams.com/modlib/libhtml/diet.htm

AMPL Model

```
# diet.mod
set NUTR;
set FOOD;

param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max { i in FOOD} >= f_min[i];
param n_min { NUTR } >= 0;
param n_max {j in NUTR } >= n_min[j];
param amt {NUTR,FOOD} >= 0;

var Buy { i in FOOD} >= f_min[i], <= f_max[i]

minimize total_cost: sum { i in FOOD } cost [i] * Buy[i];
subject to diet { j in NUTR }:
        n_min[j] <= sum {i in FOOD} amt[i,j] * Buy[i] <= n_max[i];
```

AMPL Model

```
# diet.dat
data;

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH
      MTL SPG TUR;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
   A 700 10000
   C 700 10000
   B1 700 10000
   B2 700 10000 ;
# %
```

```
param amt (tr):
           A C B1 B2 :=
   BEEF 60 20 10 15
   CHK 8 0 20 20
   FISH 8 10 15 10
   HAM 40 40 35 10
   MCH 15 35 15 15
   MTL 70 30 15 15
   SPG 25 50 25 15
   TUR 60 20 15 10 ;
```

Outline

1. Course Introduction

2. Introduction
   - Resource Allocation
   - Diet Problem
   - Duality

3. Solving LP Problems
   - Fourier-Motzkin method

Duality

Resource Valuation problem: Determine the value of the raw materials on hand such that: The company must be willing to sell the raw materials should an outside firm offer to buy them at a price consistent with the market

z_i	value of a unit of raw material i
$\sum_{i=1}^{m} b_i z_i$	opportunity cost (cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal is to minimize the lost opportunity cost

$$\min \sum_{i=1}^{m} b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} z_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(2) and (3) otherwise contradicting market

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\begin{aligned} \min \sum_{i=1}^{m} y_i b_i &+ \cancel{\sum_i \rho_i b_i} \\ \sum_{i=1}^{m} y_i a_{ij} &\geq c_j, \quad j = 1 \ldots n \\ y_i &\geq 0, \quad i = 1 \ldots m \end{aligned}$$

$$\begin{aligned} \max \sum_{j=1}^{n} c_j x_j & \\ \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, \ldots, m \\ x_j &\geq 0, \quad j = 1, \ldots, n \end{aligned}$$

Outline

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
& \dots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

$$
\begin{array}{rrcl}
\max & z & = & c^T x \\
& Ax & = & b \\
& x & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Notation

- $\mathbb{N}$ natural numbers, $\mathbb{Z}$ integer numbers, $\mathbb{Q}$ rational numbers, $\mathbb{R}$ real numbers
- column vector and matrices
  scalar product: $y^T x = \sum_{i=1}^{n} y_i x_i$
- linear combination

$$\begin{aligned} x &\in \mathbb{R}^k \\ x_1, \ldots, x_k &\in \mathbb{R} \qquad x = \sum_{i=1}^{k} \lambda_i x_i \\ \lambda = (\lambda_1, \ldots, \lambda_k)^T &\in \mathbb{R}^k \end{aligned}$$

moreover:

$\lambda \geq 0$ conic combination

$\lambda^T 1 = 1 \quad (\sum_{i=1}^{k} \lambda_i = 1)$ affine combination

$\lambda \geq 0$ and $\lambda^T 1 = 1$ convex combination

Linear Programming

Abstract mathematical model:

Decision Variables (quantity) eg. x_1 units of 1, x_2 units of 2

Criterion (discriminate among solutions) eg. max profit: $6x_1 + 8x_2$

Constraints (limitations on resources) eg.

$5x_1 + 10x_2 \leq 60; 4x_1 + 4x_2 \leq 40; x_1 \geq 0; x_2 \geq 0$

$$
\begin{array}{llrcll}
\text{objective func.} & \max / \min c^T \cdot x & & & & c \in \mathbb{R}^n \\
\text{constraints} & & A \cdot x & \substack{\geq \\ = \\ \leq} & b & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\
 & & x & \geq & 0 & x \in \mathbb{R}^n, 0 \in \mathbb{R}^n
\end{array}
$$

Essential features of a Linear program:

1. continuity (later, integrality)
2. linearity $\rightsquigarrow$ proportionality + additivity
3. certainty of parameters

Notions of Computer Science

Algorithm: a finite, well-defined sequence of operations to perform a calculation

Algorithm: LargestNumber

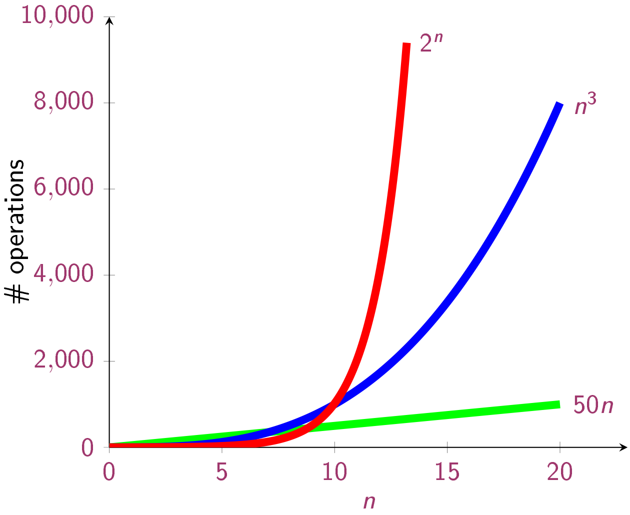
Input: A non-empty list of numbers L
Output: The largest number in the list L
largest ← L[0]
foreach each item in the list L **do**
    if the item > largest **then**
        largest ← the item
return largest

L:

2	3	5	1	8	1	4

Running time: proportional to number of operations, eg $O(n)$

Growth Functions



NP-hard problems: bad if we have to solve them, good for cryptology

History of Linear Programming (LP)

- Origins date back to Newton, Leibnitz, Lagrange, etc.
- In 1827, Fourier described a variable elimination method for systems of linear inequalities, today often called Fourier-Moutzkin elimination (Motzkin, 1937). It can be turned into an LP solver but inefficient.
- In 1932, Leontief (1905-1999) Input-Output model to represent interdependencies between branches of a national economy (1976 Nobel prize)
- In 1939, Kantorovich (1912-1986): Foundations of linear programming (Nobel prize with Koopmans on LP, 1975)
- The math subfield of Linear Programming was created by George Dantzig, John von Neumann (Princeton), and Leonid Kantorovich in the 1940s.
- In 1947, Dantzig (1914-2005) invented the (primal) simplex algorithm working for the US Air Force at the Pentagon. (program=plan)

History of LP (cntd)

- In 1954, Lemke: dual simplex algorithm, In 1954, Dantzig and Orchard Hays: revised simplex algorithm
- In 1958, Integer Programming was born with cutting planes by Gomory and branch and bound
- In 1970, Victor Klee and George Minty created an example that showed that the classical simplex algorithm has exponential worst-case behaviour.
- In 1979, L. Khachain found a new efficient algorithm for linear programming. It was terribly slow. (Ellipsoid method)
- In 1984, Karmarkar discovered yet another new efficient algorithm for linear programming. It proved to be a strong competitor for the simplex method. (Interior point method)

History of Optimization

- In 1951, Nonlinear Programming began with the Karush-Kuhn-Tucker Conditions
- In 1952, Commercial Applications and Software began
- In 1950s, Network Flow Theory began with the work of Ford and Fulkerson.
- In 1955, Stochastic Programming began
- In 1958, Integer Programming began by R. E. Gomory.
- In 1962, Complementary Pivot Theory

Outline

Fourier Motzkin elimination method

Has $Ax \leq b$ a solution? (Assumption: $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^n$)
Idea:

1. transform the system into another by eliminating some variables such that the two systems have the same solutions over the remaining variables.
2. reduce to a system of constant inequalities that can be easily decided

Let x_r be the variable to eliminate
Let $M = \{1 \ldots m\}$ index the constraints
For a variable j let partition the rows of the matrix in

$$\begin{aligned} N &= \{i \in M \mid a_{ij} < 0\} \\ Z &= \{i \in M \mid a_{ij} = 0\} \\ P &= \{i \in M \mid a_{ij} > 0\} \end{aligned}$$

$$\begin{cases} x_r \geq b_{ir} - \sum_{k=1}^{r-1} a_{ik} x_k \\ x_r \leq b_{ir} - \sum_{k=1}^{r-1} a_{ik} x_k \\ \text{all other constraints}(i \in Z) \end{cases} \qquad \begin{cases} x_r \geq A_i(x_1, \dots, x_{r-1}), & i \in N \\ x_r \leq B_i(x_1, \dots, x_{r-1}), & i \in P \\ \text{all other constraints}(i \in Z) \end{cases}$$

Hence the original system is equivalent to

$$\begin{cases} \max\{A_i(x_1, \dots, x_{r-1}), i \in N\} \leq x_r \leq \min\{B_i(x_1, \dots, x_{r-1}), i \in P\} \\ \text{all other constraints}(i \in Z) \end{cases}$$

which is equivalent to

$$\begin{cases} A_i(x_1, \dots, x_{r-1}) \leq B_j(x_1, \dots, x_{r-1}) & i \in N, j \in P \\ \text{all other constraints}(i \in Z) \end{cases}$$

we eliminated x_r but:

$$\begin{cases} |N| \cdot |P| \text{ inequalities} \\ |Z| \text{ inequalities} \end{cases}$$

after d iterations if $|P| = |N| = n/2$ exponential growth: $1/4(n/2)^{2^d}$

Example

$$\begin{array}{rcrcr} -7x_1 & + & 6x_2 & \leq & 25 \\ x_1 & - & 5x_2 & \leq & 1 \\ x_1 & & & \leq & 7 \\ -x_1 & + & 2x_2 & \leq & 12 \\ -x_1 & - & 3x_2 & \leq & 1 \\ 2x_1 & - & x_2 & \leq & 10 \end{array}$$

x_2 variable to eliminate
$N = \{2, 5, 6\}, Z = \{3\}, P = \{1, 4\}$
$|Z \cup (N \times P)| = 7$ constraints

Summary