DM545

Linear and Integer Programming

Lecture 10
Well Solved Problems
Network Flows

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Well Solved Problems

2. (Minimum Cost) Network Flows

Outline

1. Well Solved Problems

2. (Minimum Cost) Network Flows

Theoretical analysis to prove results about

- strength of certain inequalities that are facet defining
  2 ways
- descriptions of convex hull of some discrete $X \subseteq \mathbb{Z}^*$
  several ways, we see one next

Example

Example: Let $X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^m x_i \leq my, x_i \leq 1 \text{ for } i = 1, \ldots, m$
and $P = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$.
Polyhedron P describes $\text{conv}(X)$

Totally Unimodular Matrices

When the LP solution to this problem

$$IP : \max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n_+\}$$

with all data integer will have integer solution?

$$\left[\begin{array}{c:c:c:c} A & I & 0 & b \\ \hdashline c & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} & A_B x_B + A_N x_N = b \\ & A_B x_B = b, \\ & A_B \; m \times m \text{ non singular matrix} \end{aligned}$$

Cramer's rule for solving systems of equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad x = B^{-1} b = \frac{B^{adj} b}{\det(B)}$$

Definition

- A square integer matrix B is called unimodular (UM) if $\det(B) = \pm 1$
- An integer matrix A is called totally unimodular (TUM) if every square, nonsingular submatrix of A is UM

Proposition

- *If A is TUM then all vertices of $R_1(A) = \{x : Ax = b, x \geq 0\}$ are integer if b is integer*
- *If A is TUM then all vertices of $R_2(A) = \{x : Ax \leq b, x \geq 0\}$ are integer if b is integer.*

Proof: if A is TUM then $\left[A \mid I \right]$ is TUM

Any square, nonsingular submatrix C of $\left[A \mid I \right]$ can be written as

$$C = \left[\begin{array}{c|c} B & 0 \\ \hline D & I_k \end{array} \right]$$

where B is square submatrix of A. Hence $\det(C) = \det(B) = \pm 1$

Proposition

The transpose matrix A^T of a TUM matrix A is also TUM.

Theorem (Sufficient condition)

An integer matrix A with is TUM if

1. $a_{ij} \in \{0, -1, +1\}$ *for all* i, j
2. *each column contains at most two non-zero coefficients* ($\sum_{i=1}^{m} |a_{ij}| \leq 2$)
3. *if the rows can be partitioned into two sets* I_1, I_2 *such that:*
   - *if a column has 2 entries of same sign, their rows are in different sets*
   - *if a column has 2 entries of different signs, their rows are in the same set*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: by induction

Basis: one matrix of one element is TUM

Induction: let C be of size k.
If C has column with all 0s then it is singular.
If a column with only one 1 then expand on that by induction
If 2 non-zero in each column then

$$\forall j : \sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij}$$

but then linear combination of rows and $\det(C) = 0$

Other matrices with integrality property:

- TUM
- Balanced matrices
- Perfect matrices
- Integer vertices

Defined in terms of forbidden substructures that represent fractionating possibilities.

Proposition

A is always TUM if it comes from

- *node-edge incidence matrix of undirected bipartite graphs (ie, no odd cycles)* ($I_1 = U, I_2 = V, B = (U, V, E)$)
- *node-arc incidence matrix of directed graphs* ($I_2 = \emptyset$)

Eg: Shortest path, max flow, min cost flow, bipartite weighted matching

Outline

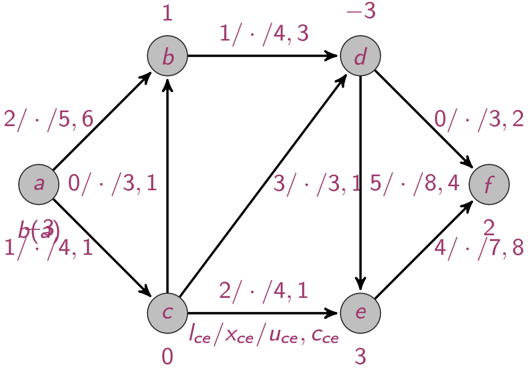
1. Well Solved Problems

2. (Minimum Cost) Network Flows

Terminology

Network:
- directed graph $D = (V, A)$
- arc, directed link, from tail to head
- lower bound $l_{ij} > 0, \forall ij \in A$, capacity $u_{ij} \geq l_{ij}, \forall ij \in A$
- cost c_{ij}, linear variation (if $ij \notin A$ then $l_{ij} = u_{ij} = 0, c_{ij} = 0$)
- balance vector $b(i)$, $b(i) < 0$ supply node (source), $b(i) > 0$ demand node (sink, tank), $b(i) = 0$ transhipment node (assumption $\sum_i b(i) = 0$)

$N = (V, A, l, u, b, c)$



Network Flows

Flow $x : A \to \mathbb{R}$

balance vector of x: $b_x(v) = \sum_{uv \in A} x_{uv} - \sum_{vw \in A} x_{vw}, \forall v \in V$

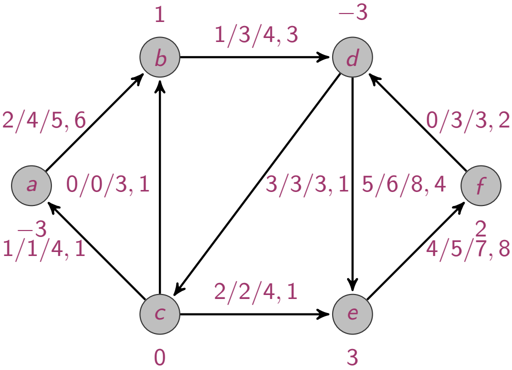
$$b_x(v) \begin{cases} > 0 & \text{sink/target/tank} \\ < 0 & \text{source} \\ = 0 & \text{balanced} \end{cases}$$

(generalizes the concept of path with $b_x(v) = \{0, 1, -1\}$)

feasible	$l_{ij} \leq x_{ij} \leq u_{ij}$, $b_x(i) = b(i)$
cost	$c^T x = \sum_{ij \in A} c_{ij} x_{ij}$ (varies linearly with x)

If iji is a 2-cycle and all $l_{ij} = 0$, then at least one of x_{ij} and x_{ji} is zero.

Example



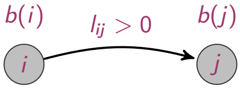
Feasible flow of cost 109

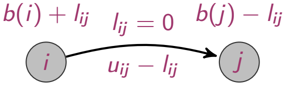
Reductions/Transformations

Lower bounds

Let $N = (V, A, l, u, b, c)$

$$
\begin{aligned}
N' &= (V, A, l', u', b') \\
b'(i) &= b(i) + l_{ij} \\
b'(j) &= b(j) - l_{ij} \\
u'_{ij} &= u_{ij} - l_{ij} \\
l'_{ij} &= 0
\end{aligned}
$$





$c^T x$

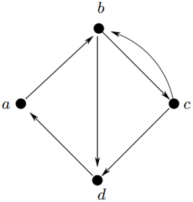
$$c^T x' + \sum_{ij \in A} c_{ij} l_{ij}$$

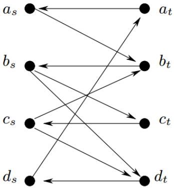
Undirected arcs

Vertex splitting

If there are bounds and costs of flow passing thorugh vertices where $b(v) = 0$ (used to ensure that a node is visited):

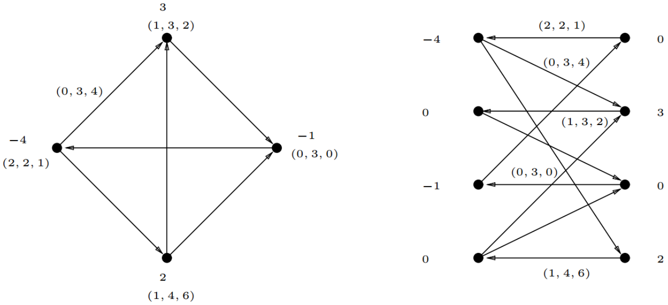
$N = (V, A, l, u, c, l^*, u^*, c^*)$





From D to D_{ST} as follows:

$$\begin{aligned} &\forall v \in V &&\rightsquigarrow v_s, v_t \in V(D_{ST}) \text{ and } v_t v_s \in A(D_{ST}) \\ &\forall xy \in A(D) &&\rightsquigarrow x_s y_t \in A(D_{ST}) \end{aligned}$$

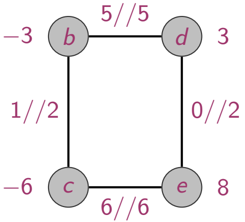


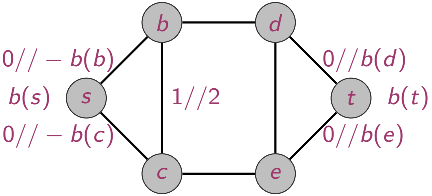
$$\forall xy \in A \text{ and } x_s y_t \in A_{ST} \rightsquigarrow h'(x_s y_t) = h(x,y),\ h \in \{l, u, c\}$$
$$\forall v \in V \text{ and } v_t v_s \in A_{ST} \rightsquigarrow h'(v_t, v_s) = h^*(v),\ h^* \in \{l^*, u^*, c^*\}$$

If $b(v) = 0$, then $b'(v_s) = b'(v_t) = 0$
If $b(v) < 0$, then $b'(v_t) = 0$ and $b'(v_s) = b(v)$
If $b(v) > 0$, then $b'(v_t) = b(v)$ and $b'(v_s) = 0$

(s,t)-**flow:**

$$b_x(v) = \begin{cases} -k & \text{if } v = s \\ k & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}, \qquad |x| = b_x(s)$$





$$b(s) = \sum_{v:v(v)<0} b(v) = -M$$
$$b(t) = \sum_{v:v(v)>0} b(v) = M$$

$\exists$ feasible flow in N $\iff$ $\exists$ (s,t)-flow in N_{st} with $|x| = M$
$\iff$ max flow in N_{st} is M

Minimum Cost Network Flows

Find cheapest flow through a network in order to satisfy demands at certain nodes from available supplier nodes.

Variables:

$$x_{ij} \in \mathbb{R}_0^+$$

Objective:

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\begin{aligned} \min\ & c^T x \\ & Nx = b \\ & 0 \leq x \leq u \end{aligned}$$

Constraints: mass balance + flow bounds

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b(i) \quad \forall i \in V$$

N node arc incidence matrix

$$0 \leq x_{ij} \leq u_{ij}$$

(assumption: all values are integer, we can multiply if rational)

$$
\begin{array}{r|cccccccc}
 & x_{e_1} & x_{e_2} & \cdots & x_{ij} & \cdots & x_{e_m} & & \\
 & c_{e_1} & c_{e_2} & \cdots & c_{ij} & \cdots & c_{e_m} & & \\
\hline
1 & -1 & . & \cdots & . & \cdots & . & = & b_1 \\
2 & . & . & \cdots & . & \cdots & . & = & b_2 \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
i & 1 & . & \cdots & -1 & \cdots & . & = & b_i \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
j & . & . & \cdots & 1 & \cdots & . & = & b_j \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
n & . & . & \cdots & . & \cdots & . & = & b_j \\
\hline
e_1 & -1 & & & & & & \geq & -u_1 \\
e_2 & & -1 & & & & & \geq & -u_2 \\
\vdots & \vdots & \ddots & & & & & \geq & \vdots \\
(i,j) & & & & -1 & & & \geq & -u_{ij} \\
\vdots & \vdots & \ddots & & & & & \geq & \vdots \\
e_m & & & & & & -1 & \geq & -u_m
\end{array}
$$

Special cases

Shortest path problem path of minimum cost from s to t with costs $\lesseqgtr 0$
$b(s) = -1, b(t) = 1, b(i) = 0$
if to any other node? $b(s) = -(n-1), b(i) = 1, u_{ij} = n-1$

Max flow problem incur no cost but restricted by bounds
steady state flow from s to t
$b(i) = 0 \ \forall i \in V, \qquad c_{ij} = 0 \ \forall ij \in A \qquad ts \in A$
$c_{ts} = -1, \qquad u_{ts} = \infty$

Assignment problem min weighted bipartite matching,
$|V_1| = |V_2|, A \subseteq V_1 \times V_2$
c_{ij}
$b(i) = -1 \ \forall i \in V_1 \qquad b(i) = 1 \ \forall i \in V_2 \qquad u_{ij} = 1 \ \forall ij \in A$

Special cases

Transportation problem/Transhipment distribution of goods, warehouses-costumers

$|V_1| \neq |V_2|, \qquad u_{ij} = \infty \; \forall ij \in A$

$$
\begin{aligned}
\min \sum c_{ij} x_{ij} & \\
\sum_i x_{ij} &\geq b_j \;\forall j \\
\sum_j x_{ij} &\leq a_i \;\forall i \\
x_{ij} &\geq 0
\end{aligned}
$$

Min cost circulation problem $b(i) = 0 \; \forall i \in V$

Multi-commodity flow problem ship several commodities using the same network, different origin destination pairs separate mass balance constraints, share capacity constraints, min overall flow

$$\begin{aligned} \min \textstyle\sum_k c^k x^k & \\ N x^k \geq b^k & \qquad \forall k \\ \textstyle\sum_k x_{ij}^k \geq u_{ij} & \qquad \forall ij \in A \\ 0 \leq x_{ij}^k \leq u_{ij}^k & \end{aligned}$$

How does the structure of the matrix looks like? Is it still TUM?

Minimum spanning tree connected acyclic graph that spans all nodes (see next lecture)

Summary

1. Well Solved Problems

2. (Minimum Cost) Network Flows