DM545

Linear and Integer Programming

Lecture 13

Preprocessing
More IP Modelling

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Outline

1. Preprocessing

2. Modeling with IP, BIP, MIP

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Preprocessing rules

Consider $S = \{x : a_0 x_0 + \sum_{j=1}^{n} a_j x_j \leq b, l_j \leq x_j \leq u_j, j = 0..n\}$

- Bounds on variables.
  If $a_0 > 0$ then:

$$x_0 \leq \left(b - \sum_{j: a_j > 0} a_j l_j - \sum_{j: a_j < 0} a_j u_j \right) / a_0$$

  and if $a_0 < 0$ then

$$x_0 \geq \left(b - \sum_{j: a_j > 0} a_j l_j - \sum_{j: a_j < 0} a_j u_j \right) / a_0$$

- Redundancy. The constraint $\sum_{j=0}^{n} a_j x_j \leq b$ is redundant if

$$\sum_{j: a_j > 0} a_j u_j + \sum_{j: a_j < 0} a_j l_j \leq b$$

- Infeasibility: $S = \emptyset$ if (swapping lower and upper bounds from previous case)

$$\sum_{j:a_j>0} a_j l_j + \sum_{j:a_j<0} a_j u_j > b$$

- Variable fixing. For a max problem in the form

$$\max\{c^T x : Ax \leq b, l \leq x \leq u\}$$

if $\forall i = 1..m a_{ij} \geq 0, c_j < 0$ then fix $x_j = l_j$
if $\forall i = 1..m a_{ij} < 0, c_j > 0$ then fix $x_j = u_j$

- Integer variables:

$$\lceil l_j \rceil \leq x_j \leq \lfloor u_j \rfloor$$

- Binary variables. Probing: add a constraint, eg, $x_2 = 0$ and check what happens

Example

$$
\begin{aligned}
\max\ & 2x_1 + x_2 - x_3 \\
I:\ & 5x_1 - 2x_2 + 8x_3 \leq 15 \\
II:\ & 8x_1 + 3x_2 - x_3 \geq 9 \\
III:\ & x_1 + x_2 + x_3 \leq 6 \\
& 0 \leq x_1 \leq 3 \\
& 0 \leq x_2 \leq 1 \\
& x_3 \geq 1
\end{aligned}
$$

$$
\begin{aligned}
I: 5x_1 &\leq 15 + 2x_2 - 8x_3 \leq 15 + 2 \cdot \overbrace{1}^{u_2} - 8 \cdot \overbrace{1}^{l_3} = 9 && \rightsquigarrow x_1 \leq 9/5 \\
8x_3 &\leq 15 + 2x_2 - 5x_1 \leq 15 + 2 \cdot 1 - 5 \cdot 0 = 17 && \rightsquigarrow x_3 \leq 17/8 \\
2x_2 &\geq 5x_1 + 8x_3 - 15 \geq 5 \cdot 0 + 8 \cdot 1 = -7 && \rightsquigarrow x_2 \geq -7/2, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
II: 8x_1 &\geq 9 - 3x_2 + x_3 \geq 9 - 3 + = 7 && \rightsquigarrow x_1 \geq 7/8 \\
I: 8x_3 &\geq 15 + 2x_2 - 5x_1 \leq 15 + 2 - 5 \cdot 7/8 = 101/8 && \rightsquigarrow x_3 \leq 101/64
\end{aligned}
$$

$x_1 + x_2 + x_3 \leq 9/5 + 1 + 101/64 < 6$ Hence III is redundant

Example

$$\begin{aligned} \max\ & 2x_1 + x_2 - x_3 \\ I:\ & 5x_1 - 2x_2 + 8x_3 \leq 15 \\ II:\ & 8x_1 + 3x_2 - x_3 \geq 9 \\ & 7/8 \leq x_1 \leq 9/5 \\ & 0 \leq x_2 \leq 1 \\ & 1 \leq x_3 \leq 101/64 \end{aligned}$$

Increasing x_2 makes constraints satisfied $\rightsquigarrow$ $x_2 = 1$
Decreasing x_3 makes constraints satisfied $\rightsquigarrow$ $x_3 = 1$

We are left with:

$$\max\{2x_1 : 7/8 \leq x_1 \leq 9/5\}$$

Preprocessing for Set Covering/Partitioning

1. if $e_i^T A = 0$ then the ith row can never be satisfied

$$[0\ 0 \dots 1 \dots 0] \begin{bmatrix} & & \\ \hdashline 0 \dots 0 & \dots & 0 \\ \hdashline & & \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

2. if $e_i^T A = e_k$ then $x_k = 1$ in every feasible solution

$$[0\ 0 \dots 1 \dots 0] \begin{bmatrix} & & & \\ \hdashline 0 & \dots 1 & \dots & 0 \\ \hdashline & & & \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

In SPP can remove all rows t with $a_{tk} = 1$ and set $x_j = 0$ (ie, remove cols) for all cols that cover t

3. if $e_t^T A \geq e_p^T A$ then we can remove row t, row p dominates row t (by covering p we cover t)

$$
\begin{array}{c}
t \\ \\ p
\end{array}
\begin{bmatrix}
 & 1 & 1 & & 1 & \\
 & & & & & \\
 & 1 & & & 1 &
\end{bmatrix}
$$

In SPP we can remove all cols j: $a_{tj} = 1, a_{pj} = 0$

4. if $\sum_{j \in S} Ae_j = Ae_k$ and $\sum_{j \in S} c_j \leq c_k$ then we can cover the rows by Ae_k more cheaply with S and set $x_k = 0$
(Note, we cannot remove S if $\sum_{j \in S} c_j \geq c_k$)

$$
\left[\begin{array}{c|ccc|c|c|c}
 & & 1 & & & 1 & \\
 & 1 & & & & 1 & \\
 & & & 1 & & 1 & \\
 & 0 & 0 & 0 & & 0 & \\
 & 1 & & & & 1 & \\
 & 0 & 0 & 0 & & 0 &
\end{array}\right]
$$

Outline

1. Preprocessing

2. Modeling with IP, BIP, MIP

Modeling with IP, BIP, MIP

Iterate:

1. define variables
2. use variables to express objective function
3. use variables to express constraints

a. problems with discrete input/output (knapsack, factory planning)

b. problems with logical conditions

c. combinatorial problems (sequencing, allocation, transport, assignment, partitioning)

d. network problems

Variables

discrete quantities	$\in \mathbb{Z}^n$
decision variables	$\in \mathbb{B}^n$
indicator/auxiliary variables (for logical conditions)	$\in \mathbb{B}^n$
special ordered sets	$\in \mathbb{B}^n$
incidence vector of S	$\in \mathbb{B}^n$

x binary
y integer
z continuous

Logical Conditions

Linking constraints $x = 0$ if $z = 0$, $x = 1$ if $z > 0$, $\quad z \in \mathbb{R}, x \in \mathbb{B}$

$$z > 0 \Longrightarrow x = 1 \Longrightarrow z - Mx \leq 0$$
$$x = 1 \Longrightarrow z > m \Longrightarrow z - mx \leq 0$$

Logical conditions and $0 - 1$ variables :

$$\begin{array}{ll}
X_1 \vee X_2 & \Longleftrightarrow x_1 + x_2 \geq 1 \\
X_1 \wedge X_2 & \Longleftrightarrow x_1 = 1, x_2 = 1 \\
\neg X_1 & \Longleftrightarrow x_1 = 0 \text{ or } (1 - x_1 = 1) \\
X_1 \rightarrow X_2 & \Longleftrightarrow x_1 - x_2 \leq 0 \\
X_1 \leftrightarrow X_2 & \Longleftrightarrow x_1 - x_2 = 0
\end{array}$$

Examples

- $(X_A \vee X_B) \rightarrow (X_C \vee X_D \vee X_E)$

$$\begin{aligned} &x_A + x_B \geq 1 \\ &x_A + x_B \geq 1 \Longrightarrow x = 1 \\ &x_A + x_B - 2x \leq 0 \end{aligned} \qquad \begin{aligned} &x_C + x_D + x_E \geq 1 \\ &x = 1 \Longrightarrow x_C + x_D + x_E \geq 1 \\ &x_C + x_D + x_E \geq x \end{aligned}$$

- Disjunctive constraints (encountered earlier)

- Constraint: $x_1 x_2 = 0$

  1) replace $x_1 x_2$ by x_3
  2) $x_3 = 1 \iff x_1 = 1, x_2 = 1$

$$\begin{aligned} -x_1 \quad\quad\; + x_3 &\leq 0 \\ - x_2 + x_3 &\leq 0 \\ x_1 + x_2 - x_3 &\leq 1 \end{aligned}$$

- $z \cdot x, \qquad z \in \mathbb{R}, x \in \mathbb{B}$

  1) replace zx by z_1
  2) impose:

$$\begin{aligned} x = 0 &\iff z_1 = 0 \\ x = 1 &\iff z_1 = z \end{aligned}$$

$$\begin{aligned} z_1 - Mx \qquad &\leq 0 \\ -z + z_1 \qquad &\leq 0 \\ z - z_1 + Mx &\leq M \end{aligned}$$

- Special ordered sets of type 1/2 (for continuous or integer vars):
  SOS1: set of vars within which exactly one must be non-zero
  SOS2: set of vars within which at most two can be non-zero. The two variables must be adjacent in the ordering

- separable programming and piecewise linear functions (next 5 slides)

Separable Programming

- Separable functions: sum of functions of single variables:

$$x_1^2 + 2x_2 + e^{x^3} \quad \text{YES}$$

$$x_1 x_2 + \frac{x_2}{x_1 + 1} + x_3 \quad \text{NO}$$

(actually, some non-separable can also be made separable:

1. $x_1 x_2$ by y
2. relate y to x_1 and x_2 by:

$$\log y = \log x_1 + \log x_2$$

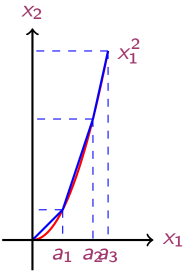
needs care if x_1 and x_2 close to zero.)

- non-linear separable functions can be approximated by piecewise linear functions
(valid for both constraints and objective functions)

Convex Non-linear Functions

- We can model convex non-linear functions by piece-wise linear functions and LP

$$
\begin{array}{rl}
\min & x_1^2 - 4x_1 - 2x_2 \\
 & x_1 + x_2 \leq 4 \\
 & 2x_1 + x_2 \leq 5 \\
 & -x_1 + 4x_2 \geq 2 \\
 & x_1, x_2 \geq 0
\end{array}
$$



- LP Formulation

$$
\begin{array}{l}
x = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \\
y = \lambda_0 f(a_0) + \lambda_1 f(a_1) + \lambda_2 f(a_2) + \lambda_3 f(a_3) \\
\sum_{i=0}^{3} \lambda_i = 1 \\
\lambda_i \geq 0 \qquad i = 0, \ldots, 3 \\
\text{at most two adjacent } \lambda_i \text{ can be non zero} \qquad (*)
\end{array}
$$

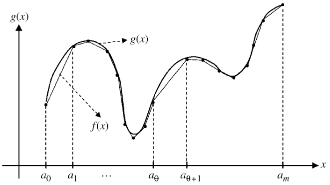
- To model (*) which are SOS2 we would need binary indicator variables and hence BIP as in next slide.

- However since the problem is convex, an optimal solution lies on the borders of the functions and hence we can skip introducing the binary variables and relax (*)

Non-convex Functions
Piece-wise Linear Functions

- non-convex functions require indicator variables and IP formulation

$$g(x) = \sum_j g_j(x) \quad g_j \text{ non linear}$$



$f(x)$ is a piecewise linear function of $g(x)$

- approximated by $f(x)$ piecewise linear in the disjoint intervals $[a_i, b_i]$
- convex hull formulation (convex combination of points)

$$\bigcup_{i \in I} \begin{pmatrix} x = & \lambda a_i + \mu b_i \\ y = & \lambda f(a_i) + \mu f(b_i) \\ \lambda + \mu = 1 & \lambda, \mu \geq 0 \end{pmatrix}$$

Remember how we modeled disjunctive polyhedra...

(cntd)

- using indicator variables δs we obtain the BIP formulation:

$$
\begin{array}{l}
x = \sum_{i \in I} (\lambda_i a_i + \mu_i b_i) \\
y = \sum_{i \in I} (\lambda_i f(a_i) + \mu_i f(b_i)) \\
\lambda_i + \mu_i = \delta_i \quad \forall i \in I \\
\sum_{i \in I} \delta_i = 1 \\
\lambda_i, \mu_i \geq 0 \quad \forall i \in I \\
\delta_i \in \{0, 1\} \quad \forall i \in I
\end{array}
$$

the δs are SOS1.

Good/Bad Models

- Number of variables: sometimes it may be advantageous increasing if they are used in search tree.

  $0-1$ var have specialized algorithms for preprocessing and for branch and bound. Hence a large number solved efficiently. Good using.

  Binary expansion:

$$0 \leq y \leq u$$

$$y = x_0 + 2x_1 + 4x_2 + 8x_3 + \ldots + 2^r x_r \qquad r = \log_2 u$$

- Making explicit good variables for branching:

$$\sum_j a_j y_j \leq b$$

$$\sum_j a_j x_j + u = b$$

  u may be a good variable to branch (u is relaxed in LP but must be integer as well)

- Symmetry breaking:
  Eg machine maintenance (see task 4, ex 3, sheet 5) $y_j \in \mathbb{Z}$ vs $x_j \in \mathbb{B}$

- Difficulty of LP models depends on number of constraints:

$$\min \sum_t |a_t z_t - b_t| \qquad \max \sum_t z'_t \qquad \max \sum_t z^+_t - z^-_t$$

$$z'_t \geq a_t z_t - b_1 \qquad z^+_t - z^-_t = a_t z_t - b_t$$

$$z'_t \geq b_t - a_t z_t$$

more variables but less constraints

- With IP it might be instead better increasing the number of constraints.

- Make M as small as possible in IP (reduces feasible region possibly fitting it to convex hull.

Practical Tips

- Units of measure: check them!
  all data should be scaled to stay in $0.1 - 10$
  some software do this automatically
- Write few line of text describing what the equations express and which are the variables, give examples on the problem modeled.
- Try the model on small simple example that can be checked by hand.
- Be diffident of infeasibility and unboundedness, double check.
- Estimate the potential size.
  If IP problem large and no structure then it might be hard.
  If TUM then solvable with very large size
  If other structure, eg, packing, covering also solvable with large size
- Check the output of the solver and understand what is happening
- If all fail resort to heuristics