DM545

Linear and Integer Programming

Lecture 4

Initialization and Duality

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Outline

1. Initialization

2. Duality
   - Derivation and Motivation
   - Theory

Simplex: Exception Handling, Overview

Handling exceptions in the Simplex Method

1. Unboundedness
2. More than one solution
3. Degeneracies
   - benign
   - cycling
4. Infeasible starting
   Phase I + Phase II

a. $F = \emptyset$

b. $F \neq \emptyset$ and $\exists$ solution
   i) one solution
   ii) infinite solution

c. $F \neq \emptyset$ and $\nexists$ solution

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Initial Infeasibility

$$\begin{array}{rrrrrl} \max & x_1 & - & x_2 & & \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rrrrrrrl} \max & x_1 & - & x_2 & & & & \\ & x_1 & + & x_2 & + x_3 & & = & 2 \\ & 2x_1 & + & 2x_2 & & - x_4 & = & 5 \\ & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

- Initial tableau

```
|    | x1 | x2 | x3 | x4 | -z | b |
|----+----+----+----+----+----+---|
| x3 |  1 |  1 |  1 |  0 |  0 | 2 |
| x4 |  2 |  2 |  0 | -1 |  0 | 5 |
|----+----+----+----+----+----+---|
|    |  1 | -1 |  0 |  0 |  1 | 0 |
```

$\leadsto$ we do not have an initial basic feasible solution!!

In general finding any feasible solution is difficult as finding an optimal solution, otherwise we could do binary search

Auxiliary Problem (I Phase of Simplex)

We introduce auxiliary variables:

$$
\begin{array}{rrrrrrl}
w^* = \max & -x_5 \equiv \min x_5 \\
& x_1 + & x_2 & + x_3 & & & = 2 \\
& 2x_1 + & 2x_2 & & - x_4 & + x_5 & = 5 \\
& & & & & x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{array}
$$

if $w^* = 0$ then $x_5 = 0$ and the two problems are equivalent
if $w^* > 0$ then not possible to set x_5 to zero.

- Initial tableau

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	5
z	1	-1	0	0	0	1	0	0
w	0	0	0	0	-1	0	1	0

Keep z always in basis

- we reach a canonical form simply by letting x_5 enter the basis:

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	5
z	1	-1	0	0	0	1	0	0
IV+II	2	2	0	-1	0	0	1	5

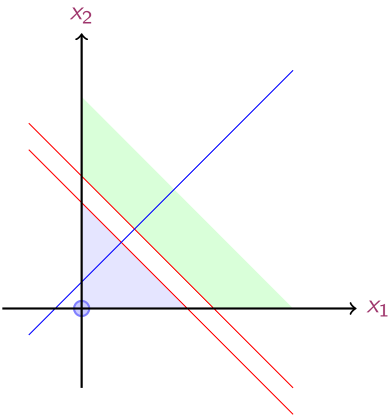
now we have a basic feasible solution!

- x_1 enters, x_3 leaves

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
II-2I'	0	0	-2	-1	1	0	0	1
III-I'	0	-2	-1	0	0	1	0	-2
IV-2I'	0	0	-2	-1	0	0	1	1

$w^* = -1$ then no solution with $X_5 = 0$ exists then no feasible solution to initial problem

$$\begin{array}{rcrcl} \max & x_1 & - & x_2 & \\ & x_1 & + & x_2 & \leq 2 \\ & 2x_1 & + & 2x_2 & \geq 5 \\ & & & x_1, x_2 & \geq 0 \end{array}$$



Initial Infeasibility - Another Example

$$\begin{array}{rrrl} \max & x_1 - x_2 & & \\ & x_1 + x_2 & \leq & 2 \\ & 2x_1 + 2x_2 & \geq & 2 \\ & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rrrl} \max & x_1 - x_2 & & \\ & x_1 + x_2 + x_3 & = & 2 \\ & 2x_1 + 2x_2 - x_4 & = & 2 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Auxiliary problem (I phase):

$$\begin{array}{rrl} w = \max -x_5 \equiv \min x_5 & & \\ x_1 + x_2 + x_3 & = & 2 \\ 2x_1 + 2x_2 - x_4 + x_5 & = & 2 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

- Initial tableau

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
z	1	-1	0	0	0	1	0	0
w	0	0	0	0	-1	0	1	0

$\leadsto$ we do not have an initial basic feasible solution.

- set in canonical form:

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
z	1	-1	0	0	0	1	0	0
IV+II	2	2	0	-1	0	0	1	2

- x_1 enters, x_5 leaves

	x1	x2	x3	x4	x5	-z	-w	b
	0	0	1	1/2	-1/2	0	0	1
	1	1	0	-1/2	1/2	0	0	1
z	0	-2	0	1/2	-1/2	1	0	-1
w	0	0	0	0	-1	0	1	0

$w^* = 0$ hence $x_5 = 0$ we have a starting feasible solution for the initial problem.

- (II phase) We keep only what we need:

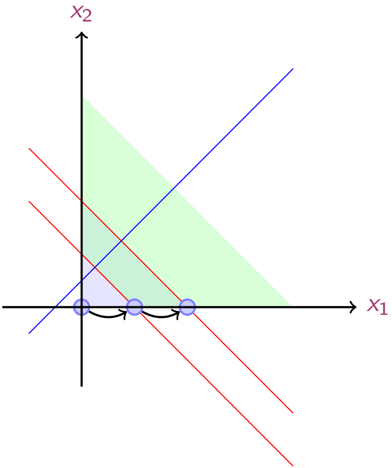
	x1	x2	x3	x4	-z	b
	0	0	1	1/2	0	1
	1	1	0	-1/2	0	1
z	0	-2	0	1/2	1	-1

-

	x1	x2	x3	x4	-z	b
	0	0	2	1	0	2
	1	1	1	0	0	2
z	0	-2	-1	0	1	-2

Optimal solution: $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 2,\ z = 2$.

$$\begin{array}{rrcrcl} \max & x_1 & - & x_2 & & \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$



In Dictionary Form

$$\begin{array}{rrrrrl} \max & x_1 & - & x_2 & & \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rcrcrcr} x_3 & = & 2 & - & x_1 & - & x_2 \\ x_4 & = & -5 & + & 2x_1 & + & 2x_2 \\ \hline z & = & x_1 & + & x_2 & & \end{array}$$

sol. infeasible

We introduce corrections of infeasibility

$$\begin{array}{rrrrrrrl} \max & -x_0 \equiv \min & x_0 & & & & & \\ & x_1 & + & x_2 & - & x_0 & \leq & 2 \\ & 2x_1 & + & 2x_2 & - & x_0 & \geq & 5 \\ & & & x_1, x_2, x_0 & & & \geq & 0 \end{array}$$

$$\begin{array}{rcrcrcrcr} x_3 & = & 2 & - & x_1 & - & x_2 & + & x_0 \\ x_4 & = & -5 & + & 2x_1 & + & 2x_2 & + & x_0 \\ \hline z & = & & & & & & - & x_0 \end{array}$$

It is still infeasible but it can be made feasible by letting x_0 enter the basis
which variable should leave?
the most infeasible: the var with the b term whose negative value has the largest magnitude

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Dual Problem

A dual variable y_i associated to each constraint:

Primal problem:

$$\begin{aligned} \max \quad z &= c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= b^T y \\ Ay &\geq c \\ y &\geq 0 \end{aligned}$$

Bounding approach

$$\begin{array}{rrrrl} \max & 4x_1 + & x_2 + & 3x_3 & \\ & x_1 + & 4x_2 & & \leq 1 \\ & 3x_1 + & x_2 + & x_3 & \leq 3 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

a feasible solution is a lower bound but how good?
By tentatives:

$$(x_1, x_2, x_3) = (1, 0, 0) \rightsquigarrow z^* \geq 4$$
$$(x_1, x_2, x_3) = (0, 0, 3) \rightsquigarrow z^* \geq 9$$

What about upper bounds?

$$\begin{array}{rl} 2 \cdot (\quad x_1 + 4x_2 \qquad) & \leq 2 \cdot 1 \\ + 3 \cdot (\; 3x_1 + x_2 + x_3) & \leq 3 \cdot 3 \\ \hline 11x_1 + 11x_2 + 3x_3 & \leq 11 \end{array}$$

$$\begin{array}{ccccc} 4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 11x_2 + 3x_3 & \leq & 11 \\ c^T x & \leq & y^T A x & \leq & y^T b \end{array}$$

Hence $z^* \leq 11$. Is this the best upper bound we can find?

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$
\begin{array}{rcl}
y_1 \cdot (\quad x_1 + 4x_2 \qquad\qquad) & \leq & y_1(1) \\
+y_2 \cdot (\ 3x_1 + \ x_2 + \qquad x_3) & \leq & y_2(3) \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 & \leq & y_1 + 3y_2
\end{array}
$$

Coefficients

$$
\begin{array}{rcrcl}
y_1 & + & 3y_2 & \geq & 4 \\
4y_1 & + & y_2 & \geq & 1 \\
 & & y_2 & \geq & 3
\end{array}
$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2$ then to attain the best upper bound:

$$
\begin{array}{lrcrcl}
\min & y_1 & + & 3y_2 & & \\
 & y_1 & + & 3y_2 & \geq & 4 \\
 & 4y_1 & + & y_2 & \geq & 1 \\
 & & & y_2 & \geq & 3 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

Multipliers Approach

$$
\begin{array}{c}
\pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1}
\end{array}
\left[\begin{array}{cccc|cccc|c|c}
a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\
\vdots & \ddots & & & & & & & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 0 & b_m \\
\hline
c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0
\end{array}\right]
$$

Working columnwise, since at optimum $\bar{c}_k \leq 0$ for all $k = 1, \dots, n + m$:

$$
\begin{cases}
\begin{array}{ccccccccc}
\pi_1 a_{11} & + & \pi_2 a_{21} & \dots + & \pi_m a_{m1} & + & \pi_{m+1} c_1 & \leq & 0 \\
\vdots & \ddots & & & & & & & \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots + & \pi_m a_{mn} & + & \pi_{m+1} c_n & \leq & 0 \\
\hline
\pi_1 a_{1,n+1}, & & \pi_2 a_{2,n+2}, & \dots & \pi_m a_{m,n+1} & & & \leq & 0 \\
\hline
& & & & & & \pi_{m+1} & = & 1 \\
\hline
\pi_1 b_1 & + & \pi_2 b_2 & \dots + & \pi_m b_m & & & (\leq & 0)
\end{array}
\end{cases}
$$

(since from the last row $z = -\pi b$ and we want to maximize z then we would $\min(-\pi b)$ or equivalently $\max \pi b$)

$$
\begin{array}{lccccccc}
\max & \pi_1 b_1 & + & \pi_2 b_2 & \dots + & \pi_m b_m & & \\
 & \pi_1 a_{11} & + & \pi_2 a_{21} & \dots + & \pi_m a_{m1} & \leq & -c_1 \\
 & \vdots & \ddots & & & & & \\
 & \pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots + & \pi_m a_{mn} & \leq & -c_n \\
 & & & & & \pi_1, \pi_2, \dots \pi_m & \leq & 0
\end{array}
$$

$y = -\pi$

$$
\begin{array}{lccccccc}
\max & -y_1 b_1 & + & -y_2 b_2 & \dots + & -y_m b_m & & \\
 & -y_1 a_{11} & + & -y_2 a_{21} & \dots + & -y_m a_{m1} & \leq & -c_1 \\
 & \vdots & \ddots & & & & & \\
 & -y_1 a_{1n} & + & -y_2 a_{2n} & \dots + & -y_m a_{mn} & \leq & -c_n \\
 & & & & & -y_1, -y_2, \dots - y_m & \leq & 0
\end{array}
$$

$$
\begin{array}{lrcl}
\min & w & = & b^T y \\
 & A^T y & \geq & c \\
 & y & \geq & 0
\end{array}
$$

Example

$$\begin{aligned} \max\ 6x_1 &+ 8x_2 \\ 5x_1 &+ 10x_2 \leq 60 \\ 4x_1 &+ 4x_2 \leq 40 \\ &x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{cases} 5\pi_1 + 4\pi_2 + 6\pi_3 \leq 0 \\ 10\pi_1 + 4\pi_2 + 8\pi_3 \leq 0 \\ 1\pi_1 + 0\pi_2 + 0\pi_3 \leq 0 \\ 0\pi_1 + 1\pi_2 + 0\pi_3 \leq 0 \\ 0\pi_1 + 0\pi_2 + 1\pi_3 = 1 \\ 60\pi_1 + 40\pi_2 \end{cases}$$

$$\begin{aligned} y_1 &= -\pi_1 \geq 0 \\ y_2 &= -\pi_2 \geq 0 \end{aligned}$$

...

Duality Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

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Symmetry

The dual of the dual is the primal:

Primal problem:

$$\begin{aligned} \max \quad z &= c^T x \\ Ax &\le b \\ x &\ge 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= b^T y \\ Ay &\ge c \\ y &\ge 0 \end{aligned}$$

Let's put the dual in the usual form

Dual problem:

$$\begin{aligned} \min \; b^T y &\equiv -\max -b^T y \\ -Ay &\le -c \\ y &\ge 0 \end{aligned}$$

Dual of Dual:

$$\begin{aligned} -\min \; c^T x & \\ -Ax &\ge -b \\ x &\ge 0 \end{aligned}$$

Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

Theorem (Weak Duality Theorem)

Given:

$$
\begin{array}{ll}
(P) & \max\{c^T x \mid Ax \leq b, x \geq 0\} \\
(D) & \min\{b^T y \mid A^T y \geq c, y \geq 0\}
\end{array}
$$

for any feasible solution x *of (P) and any feasible solution* y *of (D):*

$$c^T x \leq b^T y$$

Proof:
From (D) $c_j \leq \sum_{i=1}^m y_i a_{ij} \forall j$ and $x_j \geq 0$.
From (P) $b_i \geq \sum_{j=1}^n a_{ij} x_i \forall j$ and $y_i \geq 0$

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m y_i a_{ij} \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_i \right) y_i \leq \sum_{i=1}^m b_i y_i$$

Strong Duality Theorem

Theorem (Strong Duality Theorem) (Gale, Kuhn, Tucker, 1951; Dantzig, Von Neumann, 1947))

Given:

$$(P)\ \max\{c^T x \mid Ax \leq b, x \geq 0\}$$
$$(D)\ \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

exactly one of the following occurs:

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution* $x^* = [x_1^*, \ldots, x_n^*]$
   (D) has feasible solution $y^* = [y_1^*, \ldots, y_m^*]$

$$c^T x^* = b^T y^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.
- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)
- The last row of the final tableau will give us

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k = z^* + \sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i=1}^{m} \bar{c}_{n+i} x_{n+i} \qquad (*)$$

$$= z^* + \bar{c}_B x_B + \bar{c}_N x_N$$

  In addition, $z^* = \sum_{j=1}^{n} c_j x_j^*$ because optimal value

- We define $\boxed{y_i^* = -\bar{c}_{n+i},}\ i = 1, 2, \ldots, m$
- We claim that $(y_1^*, y_2^*, \ldots, y_m^*)$ is a dual feasible solution satisfying $c^T x^* = b^T y^*$.

- Let's verify the claim:
  We substitute in (*) $\sum c_j x_j$ for z and $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, 2, \ldots, m$ for slack variables

$$\sum c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

$$= \left(z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

This must hold for every $(x_1, x_2, \ldots, x_n)$ hence:

$$z^* = \sum_{i=1}^{m} b_i y_i^* \qquad \Longrightarrow \; y^* \text{ satisfies } c^T x^* = b^T y^*$$

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, j = 1, 2, \ldots, n$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \ldots, n+m$:

$$\bar{c}_j \leq 0 \rightsquigarrow \quad c_j - \sum_{i=1}^{m} y_i^* a_{ij} \leq 0 \rightsquigarrow \quad \sum_{i=1}^{m} y_i^* a_{ij} \geq c_j \qquad j = 1, 2, \ldots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow \quad y_i^* = -\hat{c}_{n+i} \geq 0, \qquad i = 1, 2, \ldots, m$$

$\implies y^*$ is also dual feasible solution

Complementary Slackness Theorem

Theorem (Complementary Slackness)

A feasible solution x^* for (P)
A feasible solution y^* for (D)
Necessary and sufficient conditions for optimality of both:

$$\left(c_j - \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j^* = 0, \quad j = 1, \ldots, n$$

If $x_j^* \neq 0$ then $\sum y_i^* a_{ij} = c_j$ (no surplus)
If $\sum y_i^* a_{ij} > c_j$ then $x_j^* = 0$

Proof:

$$z^* = cx^* \leq y^* Ax^* \leq by^* = w^*$$

Hence from strong duality theorem:

$$cx^* - yAx^* = 0$$

In scalars

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be $= 0$

Duality - Summary

- Derivation:
  - Bounding Approach
  - Multipliers Approach
  - Recipe
  - Lagrangian Multipliers Approach (next time)
- Theory:
  - Symmetry
  - Weak Duality Theorem
  - Strong Duality Theorem
  - Complementary Slackness Theorem