DM545

Linear and Integer Programming

Lecture 5
Duality
Sensitivity Analysis

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Outline

Outline

Lagrangian Duality

Relaxation: if a problem is hard to solve then find an easier problem resembling the original one that provides information in terms of bounds. Then search strongest bounds.

$$\begin{array}{rl} \min 13x_1 + 6x_2 + 4x_3 + 12x_4 & \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 & = 7 \\ 3x_1 + \quad + 2x_3 + 4x_4 & = 2 \\ x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

We wish to reduce to a problem easier to solve, ie:

$$\begin{array}{rl} \min c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ x_1, x_2, \ldots, x_n & \geq 0 \end{array}$$

solvable by inspection: if $c < 0$ then $x = +\infty$, if $c \geq 0$ then $x = 0$.
measure of violation of the constraints:

$$\begin{array}{l} 7 - (2x_1 + 3x_2 + 4x_3 + 5x_4) \\ 2 - (3x_1 + \quad + 2x_3 + 4x_4) \end{array}$$

We relax these measures in the obj. function with Lagrangian multipliers y_1, y_2.
We obtain a family of problems:

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{rrrrr} & 13x_1 & + 6x_2 & + 4x_3 & + 12x_4 \\ +y_1(7- & 2x_1 & + 3x_2 & + 4x_3 & + 5x_4) \\ +y_2(2- & 3x_1 & + & + 2x_3 & + 4x_4) \end{array} \right\}$$

1. for all $y_1, y_2 \in \mathbb{R} : \mathrm{opt}(PR(y_1, y_2)) \leq \mathrm{opt}(P)$
2. $\max_{y_1, y_2 \in \mathbb{R}} \{\mathrm{opt}(PR(y_1, y_2))\} \leq \mathrm{opt}(P)$

PR is easy to solve.
(It can be also seen as a proof of the weak duality theorem)

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{ll} & (13 - 2y_2 - 3y_2)\, x_1 \\ + & (6 - 3y_1)\, x_2 \\ + & (4 - 2y_2)\, x_3 \\ + & (12 - 5y_1 - 4y_2)\, x_4 \\ + & 7y_1 + 2y_2 \end{array} \right\}$$

if coeff. of x is < 0 then bound is $-\infty$ then LB is useless

$$\begin{array}{r} (13 - 2y_2 - 3y_2) \geq 0 \\ (6 - 3y_1) \geq 0 \\ (4 - 2y_2) \geq 0 \\ (12 - 5y_1 - 4y_2) \geq 0 \end{array}$$

If they all hold then we are left with $7y_1 + 2y_2$ because all go to 0.

$$\begin{array}{lrcr} \max & 7y_1 + 2y_2 & & \\ & 2y_2 + 3y_2 & \leq & 13 \\ & 3y_1 & \leq & 6 \\ & + 2y_2 & \leq & 4 \\ & 5y_1 + 4y_2 & \leq & 12 \end{array}$$

General Formulation

$$\begin{aligned} \min \quad & z = c^T x \quad c \in \mathbb{R}^n \\ & Ax = b \qquad A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\ & x \geq 0 \qquad x \in \mathbb{R}^n \end{aligned}$$

$$\max_{y \in \mathbb{R}^m} \{ \min_{x \in \mathbb{R}^n_+} \{ cx + y(b - Ax) \} \}$$

$$\max_{y \in \mathbb{R}^m} \{ \min_{x \in \mathbb{R}^n_+} \{ (c - yA)x + yb \} \}$$

$$\begin{aligned} \max \quad & b^T y \\ & A^T y \quad \leq c \\ & y \in \mathbb{R}^m \end{aligned}$$

Outline

Dual Simplex

Dual simplex (Lemke, 1954): apply the simplex method to the dual problem and observe what happens in the primal tableaux:

- Primal works with feasible solutions towards optimality
- Dual works with optimal solutions towards feasibility

Primal simplex on primal problem:

1. pivot > 0
2. col c_j with wrong sign
3. row: $\min\left\{\frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, .., m\right\}$

Dual simplex on primal problem:

1. pivot < 0
2. row $b_i < 0$ (condition of feasibility)
3. col: $\min\left\{\left|\frac{c_j}{a_{ij}}\right| : a_{ij} < 0, j = 1, 2, .., n + m\right\}$ (least worsening solution)

It can work better in some cases than the primal.
Eg. since running time in practice between $2m$ and $3m$, then if $m = 99$ and $n = 9$ then better the dual
Dual based Phase I algorithm (Dual-primal algorithm) (see Sheet 3)

Dual Simplex

Example

Primal:

$$\begin{aligned} \max \quad & -x_1 - x_2 \\ & -2x_1 - x_2 \leq 4 \\ & -2x_1 + 4x_2 \leq -8 \\ & -x_1 + 3x_2 \leq -7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Initial tableau

```
|   | x1 | x2 | w1 | w2 | w3 | -z |  b |
|---+----+----+----+----+----+----+----|
|   | -2 | -1 |  1 |  0 |  0 |  0 |  4 |
|   | -2 |  4 |  0 |  1 |  0 |  0 | -8 |
|   | -1 |  3 |  0 |  0 |  1 |  0 | -7 |
|---+----+----+----+----+----+----+----|
|   | -1 | -1 |  0 |  0 |  0 |  1 |  0 |
```

infeasible start

- x_1 enters, w_2 leaves

Dual:

$$\begin{aligned} \min \quad & 4y_1 - 8y_2 - 7y_3 \\ & -2y_1 - 2y_2 - y_3 \geq -1 \\ & -y_1 + 4y_2 + 3y_3 \geq -1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

- Initial tableau ($\min by \equiv -\max -by$)

```
|   | y1 | y2 | y3 | z1 | z2 | -p | b |
|---+----+----+----+----+----+----+---|
|   |  2 |  2 |  1 |  1 |  0 |  0 | 1 |
|   |  1 | -4 | -3 |  0 |  1 |  0 | 1 |
|---+----+----+----+----+----+----+---|
|   | -4 |  8 |  7 |  0 |  0 |  1 | 0 |
```

feasible start (thanks to $-x_1 - x_2$)

- y_2 enters, z_1 leaves

- x_1 enters, w_2 leaves

	x1	x2	w1	w2	w3	-z	b
	0	-5	1	-1	0	0	12
	1	-2	0	-0.5	0	0	4
	0	1	0	-0.5	1	0	-3
	0	-3	0	-0.5	0	1	4

- w_2 enters, w_3 leaves (note that we kept $c_j < 0$, ie, optimality)

	x1	x2	w1	w2	w3	-z	b
	0	-7	1	0	-2	0	18
	1	-3	0	0	-1	0	7
	0	-2	0	1	-2	0	6
	0	-4	0	0	-1	1	7

- y_2 enters, z_1 leaves

	y1	y2	y3	z1	z2	-p	b
	1	1	0.5	0.5	0	0	0.5
	5	0	-1	2	1	0	3
	-4	0	3	-12	0	1	-4

- y_3 enters, y_2 leaves

	y1	y2	y3	z1	z2	-p	b
	2	2	1	1	0	0	1
	7	2	0	3	1	0	3
	-18	-6	0	-7	0	1	-7

Economic Interpretation

$$
\begin{aligned}
\max\ 5x_0 + 6x_1 + \ 8x_2 & \\
6x_0 + 5x_1 + 10x_2 &\le 60 \\
8x_0 + 4x_1 + 4x_2 &\le 40 \\
4x_0 + 5x_1 + 6x_2 &\le 50 \\
x_0, x_1, x_2 &\ge 0
\end{aligned}
$$

final tableau:

$x0$	$x1$	$x2$	$s1$	$s2$	$s3$	$-z$	b
	0	1		0			5/2
	1	0		0			7
	0	0		1			2
$-1/5$	0	0	$-1/5$	0	-1		62

- Which are the values of variables, the reduced costs, the shadow prices (or marginal price), the values of dual variables?
- If one slack variable > 0 then overcapacity
- How many products can be produced at most? at most m
- How much more expensive a product not selected should be?
  look at reduced costs: $c - \pi A > 0$
- What is the value of extra capacity of manpower? In 1+1 out 1/5+1

Game: Suppose two economic operators:

- P owns the factory and produces goods
- D is the market buying and selling raw material and resources
- D asks P to close and sell him all resources
- P considers if the offer is convenient
- D wants to spend less possible
- y are prices that D offers for the resources
- $\sum y_i b_i$ is the amount D has to pay to have all resources of P
- $\sum y_i a_{ij} \geq c_j$ total value to make j > price per unit of product
- P either sells all resources $\sum y_i a_{ij}$ or produces product j (c_j)
- without $\geq$ there would not be negotiation because P would be better off producing and selling
- at optimality the situation is indifferent (strong th.)
- resource 2 that was not totally utilized in the primal has been given value 0 in the dual. (complementary slackness th.) Plausible, since we do not use all the resource, likely to place not so much value on it.
- for product 0 $\sum y_i a_{ij} > c_j$ hence not profitable producing it. (complementary slackness th.)

Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian (to do)
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
- Dual Simplex
- Economic interpretation