DM545

Linear and Integer Programming

Lecture 9
IP Modeling
Formulations, Relaxations

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Outline

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Traveling Salesman Problem

- Find the cheapest movement for a drilling, welding, drawing, soldering arm as, for example, in a printed circuit board manufacturing process or car manufacturing process

- n locations, c_{ij} cost of travel

Variables:

$$x_{ij} = \begin{cases} 1 \\ 0 \end{cases}$$

Objective:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Constraints:

-

$$\sum_{j:j\neq i} x_{ij} = 1 \qquad \forall i = 1, \ldots, n$$

$$\sum_{i:i\neq j} x_{ij} = 1 \qquad \forall j = 1, \ldots, n$$

- cut set constraints

$$\sum_{i\in S} \sum_{j\notin S} x_{ij} \geq 1 \qquad \forall S \subset N, S \neq \emptyset$$

- subtour elimination constraints

$$\sum_{i\in S} \sum_{j\in S} x_{ij} \leq |S| - 1 \qquad \forall S \subset N, 2 \leq |S| \leq n - 1$$

Outline

Modeling Tricks

Objective function and/or constraints do not appear to be linear?

- Absolute values
- Minimize the largest function value
- Maximize the smallest function value
- Constraints include variable division
- Constraints are either/or
- A variable must take one of several candidate values

Modeling Tricks I

Minimize the largest of a number of function values:

$$\min \quad \max\{f(x_1), \ldots, f(x_n)\}$$

- Introduce an auxiliary variable X:

$$\begin{aligned} \min \quad & X \\ \text{s. t.} \quad & f(x_1) \leq X \\ & f(x_2) \leq X \end{aligned}$$

Modeling Tricks II

Constraints include variable division:

- Constraint of the form

$$\frac{a_1 x + a_2 y + a_3 z}{d_1 x + d_2 y + d_3 z} \leq b$$

  - Rearrange:

$$a_1 x + a_2 y + a_3 z \leq b(d_1 x + d_2 y + d_3 z)$$

  which gives:

$$(a_1 - bd_1)x + (a_2 - bd_2)y + (a_3 - bd_3)z \leq 0$$

III "Either/Or Constraints"

In conventional mathematical models, the solution must satisfy all constraints.
Suppose that your constraints are "either/or":

$a_1x_1 + a_2x_2 \leq b_1$ or
$d_1x_1 + d_2x_2 \leq b_2$

Introduce new variable $y \in \{0, 1\}$ and a large number M:

$a_1x_1 + a_2x_2 \leq b_1 + My$ if $y = 0$ then this is active
$d_1x_1 + d_2x_2 \leq b_2 + M(1 - y)$ if $y = 1$ then this is active

III "Either/Or Constraints"

Binary integer programming allows to model alternative choices:

- Eg: 2 feasible regions, ie, disjunctive constraints, not possible in LP.
  introduce y auxiliary binary variable and M a big number:

$$
\begin{aligned}
Ax &\leq b + My && \text{if } y = 0 \text{ then this is active} \\
A'x &\leq b' + M(1-y) && \text{if } y = 1 \text{ then this is active}
\end{aligned}
$$

IV "Either/Or Constraints"

Generally:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1m}x_m &\leq d_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2m}x_m &\leq d_2 \\
&\vdots \\
a_{m1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{Nm}x_m &\leq d_N
\end{aligned}
$$

Exactly K of the N constraints must be satisfied.
Introduce binary variables $y_1, y_2, \ldots, y_N$ and a large number M

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1m}x_m &\leq d_1 + My_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2m}x_m &\leq d_2 + My_2 \\
&\vdots \\
a_{m1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{Nm}x_m &\leq d_N + My_N
\end{aligned}
$$

$$y_1 + y_2 + \ldots y_N = N - K$$

K of the y-variables is 0, so K constraints must be satisfied

IV "Either/Or Constraints"

At least $h \leq k$ of $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, \ldots, k$ must be satisfied

introduce y_i, $i = 1, ..., k$ auxiliary binary variables

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + My_i$$

$$\sum_{i} y_i \leq k - h$$

V "Possible Constraints Values"

A constraint must take on one of N given values:

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m = d_1 \text{ or}$$
$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m = d_2 \text{ or}$$
$$\vdots$$
$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m = d_N$$

Introduce binary variables $y_1, y_2, \ldots, y_N$:

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m = d_1y_1 + d_2y_2 + \ldots d_Ny_N$$

$$y_1 + y_2 + \ldots y_N = 1$$

Outline

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Uncapacited Facility Location (UFL)

Given:

- depots $N = \{1, \ldots, n\}$
- clients $M = \{1, \ldots, m\}$
- f_j fixed cost to use depot j
- transport cost for all orders c_{ij}

Task: Which depots to open and which depots serve which client

Variables: $y_j = \begin{cases} 1 & \text{if depot open} \\ 0 & \text{otherwise} \end{cases}$, x_{ij} fraction of demand of i satisfied by j

Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

Constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, m$$

$$\sum_{i \in M} x_{ij} \leq m y_j \qquad \forall j \in N$$

Outline

Good and Ideal Formulations

Definition (Formulation)

A polyhedron $P \subseteq \mathbb{R}^{n+p}$ is a formulation for a set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if and only if $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region X.

There are infinite formulations

Definition (Convex Hull)

Given a set $X \subseteq \mathbb{Z}^n$ the convex hull of X is defined as:

$$\text{conv}(X) = \left\{ x : x = \sum_{i=1}^{t} \lambda_i x^i, \sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geq 0, \text{for } i = 1, \ldots, t, \right.$$
$$\left. \text{for all finite subsets } \{x^1, \ldots, x^t\} \text{ of } X \right\}$$

Proposition

$conv(X)$ is a polyhedron (ie, representable as $Ax \leq b$)

Proposition

Extreme points of $conv(X)$ all lie in X

Hence:

$$\max\{c^T x : x \in X\} \equiv \max\{c^T x : x \in \text{conv}(X)\}$$

However it might require exponential number of inequalities to describe $\text{conv}(X)$

What makes a formulation better than another?

$$X \subseteq \text{conv}(X) \subseteq P_1 \subset P_2$$

$$P_1 \text{ is better than } P_2$$

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, P_1 is a better formulation than P_2 if $P_1 \subset P_2$

Example

P_1 =UFL with $\sum_{i\in M} x_{ij} \leq my_j \quad \forall j \in N$
P_2 =UFL with $x_{ij} \leq y_j \quad \forall i \in M, j \in N$

$$P_2 \subset P_1$$

- $P_2 \subseteq P_1$ because summing $x_{ij} \leq y_j$ over $i \in M$ we obtain $\sum_{i\in M} x_{ij} \leq my_j$
- $P_2 \subset P_1$ because there exists a point in P_1 but not in P_2: $m = 6 = 3 \cdot 2 = k \cdot n$

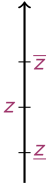
$$x_{10} = 1 \; x_{20} = 1 \; x_{30} = 1 \qquad \sum_i x_{i0} \leq 6y_0 \quad y_0 = 1/2$$
$$x_{41} = 1 \; x_{51} = 1 \; x_{61} = 1 \qquad \sum_i x_{i1} \leq 6y_1 \quad y_1 = 1/2$$

Outline

Optimality and Relaxation

$$z = \max\{c(x) : x \in X \subseteq \mathbb{Z}^n\}$$

How can we prove that x^* is optimal?
$\overline{z}$ UB
$\underline{z}$ LB
stop when $\overline{z} - \underline{z} \leq \epsilon$



- Primal bounds (here lower bounds): every feasible solution gives a LB
  may be easy or hard, heuristics
- Dual bounds (here upper bounds): Relaxations

Optimality gap:

$$\text{gap} = \frac{pb - db}{\inf\{|z|, z \in [db, pb]\}} (\cdot 100) \qquad \text{for a minimization problem}$$

(if $pb \geq 0$ and $db \geq 0$ then $\frac{pb-db}{db}$)
if $db = pb = 0$ then $\text{gap} = 0$
if no feasible sol found or $db \leq 0 \leq pb$ then the gap is not computed.

Proposition

$$\begin{array}{ll}(RP)\ z^R &= \max\{f(x) : x \in T \subseteq \mathbb{R}^n\} \textit{ is a relaxation of} \\ (IP)\ z &= \max\{c(x) : x \in X \subseteq \mathbb{R}^n\} \textit{ if :}\end{array}$$

(i) $X \subseteq T$ *or*

(ii) $f(x) \geq c(x)\ \forall x \in X$

In other terms:

$$\max_{s \in T} f(s) \geq \left\{ \begin{array}{l} \max_{s \in T} c(s) \\ \max_{s \in X} f(s) \end{array} \right\} \geq \max_{s \in X} c(s)$$

- T: candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(x) \geq c(x)$

Relaxations

How to construct relaxations?

1. $IP : \max\{c^T x : x \in P \cap \mathbb{Z}^n\}, P = \{c \in \mathbb{R}^n : Ax \leq b\}$
   $LP : \max\{c^T x : x \in P\}$
   Better formulations give better bounds ($P_1 \subseteq P_2$)

   Proposition

   (i) *If a relaxation RP is infeasible, the original problem OP is infeasible.*

   (ii) *Let x^* optimal solution for RP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is optimal for IP.*

2. Combinatorial relaxations to easy problems that can be solved rapidly
   Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$
\begin{aligned}
IP: \quad z &= \max\{c^T x : Ax \leq b, x \in X \subseteq \mathbb{Z}^n\} \\
z(u) &= \max\{c^T x + u(b - Ax) : x \in X\}
\end{aligned}
$$

$$z(u) \geq z \qquad \forall u \geq 0$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(x) : x \in X\} \qquad w = \min\{w(u) : u \in U\}$$

form a weak-dual pair if $c(x) \leq w(u)$ for all $x \in X$ and all $u \in U$.
When $z = w$ they form a strong-dual pair

Proposition

$z = \max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n_+\}$ *and* $w^{LP} = \min\{ub^T : uA \geq c, u \in \mathbb{R}^m_+\}$ *(ie, linear relaxations) form a weak-dual pair.*

Proposition

Let IP and D be weak-dual pair:

(i) *If D is unbounded, then IP is infeasible*

(ii) *If* $x^* \in X$ *and* $u^* \in U$ *satisfy* $c(x^*) = w(u^*)$ *then* x^* *is optimal for IP and* u^* *is optimal for D.*

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Examples

Weak pairs:

Matching: $z = \max\{1^T x : Ax \leq 1, x \in \mathbb{Z}_+^m\}$

V. Covering: $w = \min\{1^T y : y^T A \geq 1, y \in \mathbb{Z}_+^n\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:

Packing: $z = \max\{1^T x : Ax \leq 1, x \in \mathbb{Z}_+^n\}$

S. Covering: $w = \min\{1^T x : Ax \geq 1, x \in \mathbb{Z}_+^n\}$

Outline

Separation problem

$\max\{c^T x : x \in X\} \equiv \max\{c^T x : x \in \text{conv}(X)\}$
$X \subseteq \mathbb{Z}^n$, P a polyhedron $P \subseteq \mathbb{R}^n$ and $X = P \cap \mathbb{Z}^n$

Definition (Separation problem for a COP)

Given $x^* \in P$ is $x^* \in \text{conv}(X)$? If not find an inequality $ax \leq b$ satisfied by all points in X but violated by the point x^*.

(Farkas lemma states the existence of such an inequality.)

Properties of Easy Problems

Four properties that often go together:

Definition

(i) Efficient optimization property: $\exists$ a polynomial algorithm for $\max\{cx : x \in X \subseteq \mathbb{R}^n\}$

(ii) Strong duality property: $\exists$ strong dual D $\min\{w(u) : u \in U\}$ that allows to quickly verify optimality

(iii) Efficient separation problem: $\exists$ efficient algorithm for separation problem

(iv) Efficient convex hull property: a compact description of the convex hull is available

Example:

If explicit convex hull strong duality holds
efficient separation property (just description of $\text{conv}(X)$)

Resume

1. Modeling
   - Assignment Problem
   - Set Covering
   - Graph Problems
   - Modeling Tricks

2. Formulations
   - Uncapacited Facility Location
   - Alternative Formulations

3. Relaxations

4. Well Solved Problems