DM811

Heuristics for Combinatorial Optimization

**Lecture 5**

# Construction Heuristics, Examples: SAT

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# Outline

1. SAT

# Outline

# SAT Problem

## Satisfiability problem in propositional logic



$$
\begin{aligned}
&(x_5 \vee x_8 \vee \bar{x}_2) \wedge (x_2 \vee \bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_3 \vee \bar{x}_7) \wedge (\bar{x}_5 \vee x_3 \vee x_8) \wedge \\
&(\bar{x}_6 \vee \bar{x}_1 \vee \bar{x}_5) \wedge (x_8 \vee \bar{x}_9 \vee x_3) \wedge (x_2 \vee x_1 \vee x_3) \wedge (\bar{x}_1 \vee x_8 \vee x_4) \wedge \\
&(\bar{x}_9 \vee \bar{x}_6 \vee x_8) \wedge (x_8 \vee x_3 \vee \bar{x}_9) \wedge (x_9 \vee \bar{x}_3 \vee x_8) \wedge (x_6 \vee \bar{x}_9 \vee x_5) \wedge \\
&(x_2 \vee \bar{x}_3 \vee \bar{x}_8) \wedge (x_8 \vee \bar{x}_6 \vee \bar{x}_3) \wedge (x_8 \vee \bar{x}_3 \vee \bar{x}_1) \wedge (\bar{x}_8 \vee x_6 \vee \bar{x}_2) \wedge \\
&(x_7 \vee x_9 \vee \bar{x}_2) \wedge (x_8 \vee \bar{x}_9 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_9 \vee x_4) \wedge (x_8 \vee x_1 \vee \bar{x}_2) \wedge \\
&(x_3 \vee \bar{x}_4 \vee \bar{x}_6) \wedge (\bar{x}_1 \vee \bar{x}_7 \vee x_5) \wedge (\bar{x}_7 \vee x_1 \vee x_6) \wedge (\bar{x}_5 \vee x_4 \vee \bar{x}_6) \wedge \\
&(\bar{x}_4 \vee x_9 \vee \bar{x}_8) \wedge (x_2 \vee x_9 \vee x_1) \wedge (x_5 \vee \bar{x}_7 \vee x_1) \wedge (\bar{x}_7 \vee \bar{x}_9 \vee \bar{x}_6) \wedge \\
&(x_2 \vee x_5 \vee x_4) \wedge (x_8 \vee \bar{x}_4 \vee x_5) \wedge (x_5 \vee x_9 \vee x_3) \wedge (\bar{x}_5 \vee \bar{x}_7 \vee x_9) \wedge \\
&(x_2 \vee \bar{x}_8 \vee x_1) \wedge (\bar{x}_7 \vee x_1 \vee x_5) \wedge (x_1 \vee x_4 \vee x_3) \wedge (x_1 \vee \bar{x}_9 \vee \bar{x}_4) \wedge \\
&(x_3 \vee x_5 \vee x_6) \wedge (\bar{x}_6 \vee x_3 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee x_5 \vee x_9) \wedge (x_7 \vee \bar{x}_5 \vee \bar{x}_2) \wedge \\
&(x_4 \vee x_7 \vee x_3) \wedge (x_4 \vee \bar{x}_9 \vee \bar{x}_7) \wedge (x_5 \vee \bar{x}_1 \vee x_7) \wedge (x_5 \vee \bar{x}_1 \vee x_7) \wedge \\
&(x_6 \vee x_7 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_6 \vee \bar{x}_7) \wedge (x_6 \vee x_2 \vee x_3) \wedge (\bar{x}_8 \vee x_2 \vee x_5)
\end{aligned}
$$

Does there exist a truth assignment satisfying all clauses?

Search for a satisfying assignment (or prove none exists)

# SAT Problem

## Satisfiability problem in propositional logic

$$
\begin{aligned}
&(x_5 \lor x_8 \lor \bar{x}_2) \land (x_2 \lor \bar{x}_1 \lor \bar{x}_3) \land (\bar{x}_8 \lor \bar{x}_3 \lor \bar{x}_7) \land (\bar{x}_5 \lor x_3 \lor x_8) \land \\
&(\bar{x}_6 \lor \bar{x}_1 \lor \bar{x}_5) \land (x_8 \lor \bar{x}_9 \lor x_3) \land (x_2 \lor x_1 \lor x_3) \land (\bar{x}_1 \lor x_8 \lor x_4) \land \\
&(\bar{x}_9 \lor \bar{x}_6 \lor x_8) \land (x_8 \lor x_3 \lor \bar{x}_9) \land (x_9 \lor \bar{x}_3 \lor x_8) \land (x_6 \lor \bar{x}_9 \lor x_5) \land \\
&(x_2 \lor \bar{x}_3 \lor \bar{x}_8) \land (x_8 \lor \bar{x}_6 \lor \bar{x}_3) \land (x_8 \lor \bar{x}_3 \lor \bar{x}_1) \land (\bar{x}_8 \lor x_6 \lor \bar{x}_2) \land \\
&(x_7 \lor x_9 \lor \bar{x}_2) \land (x_8 \lor \bar{x}_9 \lor x_2) \land (\bar{x}_1 \lor \bar{x}_9 \lor x_4) \land (x_8 \lor x_1 \lor \bar{x}_2) \land \\
&(x_3 \lor \bar{x}_4 \lor \bar{x}_6) \land (\bar{x}_1 \lor \bar{x}_7 \lor x_5) \land (\bar{x}_7 \lor x_1 \lor x_6) \land (\bar{x}_5 \lor x_4 \lor \bar{x}_6) \land \\
&(\bar{x}_4 \lor x_9 \lor \bar{x}_8) \land (x_2 \lor x_9 \lor x_1) \land (x_5 \lor \bar{x}_7 \lor x_1) \land (\bar{x}_7 \lor \bar{x}_9 \lor \bar{x}_6) \land \\
&(x_2 \lor x_5 \lor x_4) \land (x_8 \lor \bar{x}_4 \lor x_5) \land (x_5 \lor x_9 \lor x_3) \land (\bar{x}_5 \lor \bar{x}_7 \lor x_9) \land \\
&(x_2 \lor \bar{x}_8 \lor x_1) \land (\bar{x}_7 \lor x_1 \lor x_5) \land (x_1 \lor x_4 \lor x_3) \land (x_1 \lor \bar{x}_9 \lor \bar{x}_4) \land \\
&(x_3 \lor x_5 \lor x_6) \land (\bar{x}_6 \lor x_3 \lor \bar{x}_9) \land (\bar{x}_7 \lor x_5 \lor x_9) \land (x_7 \lor \bar{x}_5 \lor \bar{x}_2) \land \\
&(x_4 \lor x_7 \lor x_3) \land (x_4 \lor \bar{x}_9 \lor \bar{x}_7) \land (x_5 \lor \bar{x}_1 \lor x_7) \land (x_5 \lor \bar{x}_1 \lor x_7) \land \\
&(x_6 \lor x_7 \lor \bar{x}_3) \land (\bar{x}_8 \lor \bar{x}_6 \lor \bar{x}_7) \land (x_6 \lor x_2 \lor x_3) \land (\bar{x}_8 \lor x_2 \lor x_5)
\end{aligned}
$$

Does there exist a truth assignment satisfying all clauses?

Search for a satisfying assignment (or prove none exists)

# Motivation

- From 100 variables, 200 constraints (early 90s)
  to 1,000,000 vars. and 20,000,000 cls. in 20 years.
- Applications:
  Hardware and Software Verification, Planning, Scheduling, Optimal Control, Protocol Design, Routing, Combinatorial problems, Equivalence Checking, etc.
- SAT used to solve many other problems!

# SAT Problem

## Satisfiability problem in propositional logic

Definitions:

- Formula in propositional logic: well-formed string that may contain
  - propositional variables $x_1, x_2, \ldots, x_n$;
  - truth values $\top$ ('true'), $\bot$ ('false');
  - operators $\neg$ ('not'), $\wedge$ ('and'), $\vee$ ('or');
  - parentheses (for operator nesting).
- Model (or satisfying assignment) of a formula $F$: Assignment of truth values to the variables in $F$ under which $F$ becomes true (under the usual interpretation of the logical operators)
- Formula $F$ is satisfiable iff there exists at least one model of $F$, unsatisfiable otherwise.

SAT Problem (decision problem, search variant):

- **Given:** Formula $F$ in propositional logic
- **Task:** Find an assignment of truth values to variables in $F$ that renders $F$ true, or decide that no such assignment exists.

SAT: A simple example

- **Given:** Formula $F := (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$
- **Task:** Find an assignment of truth values to variables $x_1, x_2$ that renders $F$ true, or decide that no such assignment exists.

Definitions:

- A formula is in conjunctive normal form (CNF) iff it is of the form

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{k_i} l_{ij} = (l_{11} \vee \ldots \vee l_{1k_1}) \wedge \ldots \wedge (l_{m1} \vee \ldots \vee l_{mk_m})$$

  where each literal $l_{ij}$ is a propositional variable or its negation. The disjunctions $c_i = (l_{i1} \vee \ldots \vee l_{ik_i})$ are called clauses.

- A formula is in $k$-CNF iff it is in CNF and all clauses contain exactly $k$ literals (*i.e.*, for all $i$, $k_i = k$).

- In many cases, the restriction of SAT to CNF formulae is considered.
- For every propositional formula, there is an equivalent formula in 3-CNF.

### Example:

$$
\begin{aligned}
F := \quad & \wedge (\neg x_2 \vee x_1) \\
& \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \\
& \wedge (x_1 \vee x_2) \\
& \wedge (\neg x_4 \vee x_3) \\
& \wedge (\neg x_5 \vee x_3)
\end{aligned}
$$

- $F$ is in CNF.
- Is $F$ satisfiable?
  Yes, *e.g.*, $x_1 := x_2 := \top$, $x_3 := x_4 := x_5 := \bot$ is a model of $F$.

### MAX-SAT (optimization problem)

Which is the maximal number of clauses satisfiable in a propositional logic formula $F$?

# Exercise

**Definition**

(Maximum) $K$-Satisfiability (SAT)
**Input:** A set $U$ of variables, a collection $C$ of disjunctive clauses of at most $k$ literals, where a literal is a variable or a negated variable in $U$. $k$ is a constant, $k > 2$.
**Task:** A truth assignment for $U$ or a truth assignment that maximizes the number of clauses satisfied.

1. design one or more construction heuristics for the problem
2. show how the decision version of the graph coloring problem (GCP) can be encoded in a SAT problem
3. show how the constraint satisfaction problem (CSP) can be encoded in a SAT problem
4. are the results of the two previous points proves of the NP-completeness of the CSP and GCP?
5. devise preprocessing rules, ie, polynomial time simplification rules

# Pre-processing

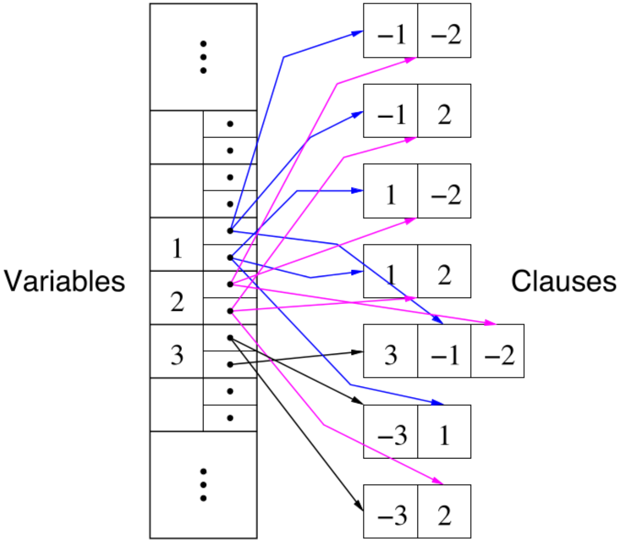
Pre-processing rules: low polynimial time procedures to decrease the size of the problem instance.

Typically applied in cascade until no rule is effective anymore.

# Examples in SAT

1. eliminate duplicate literals
2. eliminate tautologies: $x_1 \vee \neg x_1 ...$
3. eliminate subsumed clauses
4. eliminate clauses with pure literals
5. eliminate unit clauses
6. unit propagation

# Simple data structure for unit propagation



# Construction heuristics



- Variable selection heuristics
  aim: minimize the search space
  plus: could compensate a bad value selection

- Value selection heuristics
  aim: guide search towards a solution (or conflict)
  plus: could compensate a bad variable selection

- Restart strategies
  aim: avoid heavy-tail behavior [GomesSelmanCrato'97]
  plus: focus search on recent conflicts when combined with dynamic heuristics

# Variable selection heuristics



- Based on the occurrences in the (reduced) formula
  - Maximal Occurrence in clauses of Minimal Size (MOMS, Jeroslow-Wang)
- Variable State Independent Decaying Sum (VSIDS)
  - original idea (zChaff): for each conflict, increase the score of involved variables by 1, half all scores each 256 conflicts [MoskewiczMZZM2001]
  - improvement (MiniSAT): for each conflict, increase the score of involved variables by $\delta$ and increase $\delta := 1.05\delta$ [EenSörensson2003]

# Value selection heuristics

- Based on the occurrences in the (reduced) formula
  - examples: Jeroslow-Wang, Maximal Occurrence in clauses of Minimal Size (MOMS), look-aheads