DM826 – Spring 2014

Modeling and Solving Constrained Optimization Problems

Lecture 13

Symmetries

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Resume

- Modelling in IP and CP
- Global constraints
- Local consistency notions
- Filtering algorithms for global constraints
  Scheduling
- Search
- Set variables
- Symmetries

Outline

1. Symmetries in CSPs

2. Group theory

3. Avoiding symmetries
    ...by Reformulation
    ...by static Symmetry Breaking
    ...during Search
    ...by Dominance Detection (SBDD)

Symmetries

Example

$$\mathcal{P} = \langle x_i \in \{1 \dots 3\}, \forall i = 1, \dots 3; \mathcal{C} \equiv x_1 = x_2 + x_3 \rangle$$

Solutions: $(2, 1, 1)$, $(3, 1, 2)$, $(3, 2, 1)$.

Because of the symmetric nature of the plus operator, swapping the values of x_2 and x_3 gives raise to *equivalent* solutions.

- Many constraint satisfaction problem models have symmetries (some examples in a few slides)
- Breaking symmetry reduces search by avoiding to explore equivalent states (half of the search tree in the previous case)
- Inducing a preference on a (possibly singleton) subset of each solution equivalence class

Symmetry Example: Social Golfer Problem

Problem statement

Given g groups of p golf players, and w weeks. All players plays once a week, and we do not want that two player play in the same group more than once.

A possible model considers a three-dimensional matrix X_{ijk} $i \in \{1, \dots, w\}, j \in \{1, \dots, g\}, k \in \{1, \dots p\}$ of integer variables $\{1, \dots g \times p\}$ representing the player playing as k-th player during week i in group j.

Symmetry Example: Social Golfer Problem

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
week 2	0	3	6	1	4	9	2	7	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	7	9	14
week 4	0	5	14	1	10	12	2	3	8	4	7	11	6	9	13
week 5	0	7	10	1	8	13	2	4	14	3	9	12	5	6	11
week 6	0	8	9	1	5	7	2	11	13	3	10	14	4	6	12
week 7	0	11	12	1	6	14	2	5	9	3	7	13	4	8	10

Symmetry Example: Social Golfer Problem
Permuting position in group

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
week 2	0	3	6	1	4	9	2	7	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	7	9	14
week 4	0	5	14	1	10	12	2	3	8	4	7	11	6	9	13
week 5	0	7	10	1	8	13	2	4	14	3	9	12	5	6	11
week 6	0	8	9	1	5	7	2	11	13	3	10	14	4	6	12
week 7	0	11	12	1	6	14	2	5	9	3	7	13	4	8	10

Symmetry Example: Social Golfer Problem

Permuting position in group

	group 1			group 2			group 3			group 4			group 5		
week 1	2	1	0	3	4	5	6	7	8	9	10	11	12	13	14
week 2	6	3	0	1	4	9	2	7	12	5	10	13	8	11	14
week 3	13	4	0	1	3	11	2	6	10	5	8	12	7	9	14
week 4	14	5	0	1	10	12	2	3	8	4	7	11	6	9	13
week 5	10	7	0	1	8	13	2	4	14	3	9	12	5	6	11
week 6	9	8	0	1	5	7	2	11	13	3	10	14	4	6	12
week 7	12	11	0	1	6	14	2	5	9	3	7	13	4	8	10

Symmetry Example: Social Golfer Problem
Permuting groups

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
week 2	0	3	6	1	4	9	2	7	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	7	9	14
week 4	0	5	14	1	10	12	2	3	8	4	7	11	6	9	13
week 5	0	7	10	1	8	13	2	4	14	3	9	12	5	6	11
week 6	0	8	9	1	5	7	2	11	13	3	10	14	4	6	12
week 7	0	11	12	1	6	14	2	5	9	3	7	13	4	8	10

Symmetry Example: Social Golfer Problem

Permuting groups

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	9	10	11	6	7	8	3	4	5	12	13	14
week 2	0	3	6	5	10	13	2	7	12	1	4	9	8	11	14
week 3	0	4	13	5	8	12	2	6	10	1	3	11	7	9	14
week 4	0	5	14	4	7	11	2	3	8	1	10	12	6	9	13
week 5	0	7	10	3	9	12	2	4	14	1	8	13	5	6	11
week 6	0	8	9	3	10	14	2	11	13	1	5	7	4	6	12
week 7	0	11	12	3	7	13	2	5	9	1	6	14	4	8	10

Symmetry Example: Social Golfer Problem

Permuting weeks

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
week 2	0	3	6	1	4	9	2	7	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	7	9	14
week 4	0	5	14	1	10	12	2	3	8	4	7	11	6	9	13
week 5	0	7	10	1	8	13	2	4	14	3	9	12	5	6	11
week 6	0	8	9	1	5	7	2	11	13	3	10	14	4	6	12
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Symmetry Example: Social Golfer Problem

Permuting weeks

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
week 2	0	7	10	1	8	13	2	4	14	3	9	12	5	6	11
week 3	0	4	13	1	3	11	2	6	10	5	8	12	7	9	14
week 4	0	5	14	1	10	12	2	3	8	4	7	11	6	9	13
week 5	0	3	6	1	4	9	2	7	12	5	10	13	8	11	14
week 6	0	8	9	1	5	7	2	11	13	3	10	14	4	6	12
week 7	0	11	12	1	6	14	2	5	9	3	7	13	4	8	10

Symmetry Example: Social Golfer Problem
Permuting players

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
week 2	0	3	6	1	4	9	2	7	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	7	9	14
week 4	0	5	14	1	10	12	2	3	8	4	7	11	6	9	13
week 5	0	7	10	1	8	13	2	4	14	3	9	12	5	6	11
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Symmetry Example: Social Golfer Problem
Permuting players

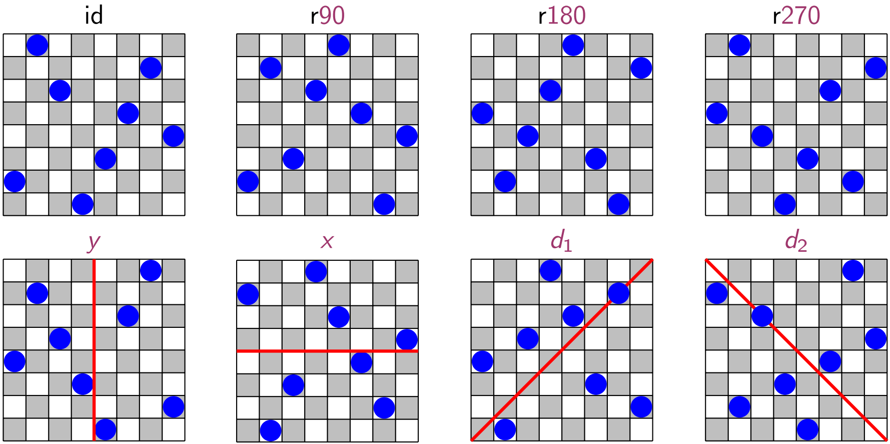
	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	9	8	7	10	11	12	13	14
week 2	0	3	6	1	4	7	2	9	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	9	7	14
week 4	0	5	14	1	10	12	2	3	8	4	9	11	6	7	13
week 5	0	9	10	1	8	13	2	4	14	3	7	12	5	6	11
week 6	0	8	7	1	5	9	2	11	13	3	10	14	4	6	12
week 7	0	11	12	1	6	14	2	5	7	3	9	13	4	8	10

Symmetry Example: Social Golfer Problem

Number of (equivalent) solutions:

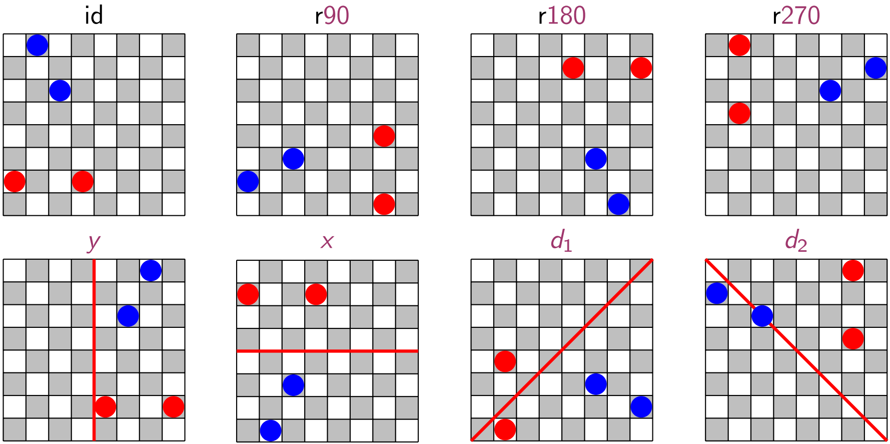
- Permuting positions: $3! = 6$
- Permuting groups: $5! = 120$
- Permuting weeks: $7! = 5040$
- Permuting players: $15! = 1,307,674,368,000$

Symmetry Example: *n*-Queens



Symmetry Example: *n*-Queens

Symmetric failure



Symmetries: general considerations

- Widespread
  - Inherent in the problem (n-Queens, chessboard)
  - Artefact of the model (Social Golfer: order of players in groups)
- Different types:
  - variable symmetry (swapping variables)
  - value symmetry (permuting values)

Types of symmetries

- Variable symmetry: permuting variables is solution invariant

$$\{x_i = v_i\} \in sol(P) \iff \{x_{\sigma(i)} = v_i\} \in sol(P)$$

- Value symmetry: permuting values is solution invariant

$$\{x_i = v_i\} \in sol(P) \iff \{x_i = \sigma(v_i)\} \in sol(P)$$

- Variable/value symmetry: both variables and values permutation is solution invariant

$$\{x_i = v_i\} \in sol(P) \iff \{x_{\sigma_1(i)} = \sigma_2(v_i)\} \in sol(P)$$

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Group basics

Group

A set G and an associated operation $\otimes$ form a group if

- G is closed under $\otimes$, i.e., $a, b \in G \Rightarrow a \otimes b \in G$
- $\otimes$ is associative, i.e., $a, b, c \in G \Rightarrow (a \otimes b) \otimes c = a \otimes (b \otimes c)$
- G has an identity ι_G, such that $a \in G \Rightarrow a \otimes \iota_G = \iota_G \otimes a = a$
- every element has an inverse, i.e., $a \in G \Rightarrow \exists a^{-1} a \otimes a^{-1} = a^{-1} \otimes a = \iota_G$

The set of permutations forms a group, together with concatenation.

Generators

Generators

A set $S \subseteq G$ is called a generator of group G iff

$$\forall g \in G \exists S' \subseteq S g = \bigotimes_{s \in S'} s$$

Generators describe groups in a compact form.
For example:

- symmetries of a square $\{r90, d1\}$
- permutations of $\{1, \ldots, n\}$: $\{(123 \ldots n), (12)\}$

Orbits

Orbits

The orbit of an element with respect to a permutation group G is

$$O_G(g) = \{\sigma(g) | \sigma \in G\}$$

The orbit of a set of elements (called also points) is defined accordingly.

Orbits are the set of elements encountered by starting from one element and moving through different permutations.

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How to avoid symmetry

Never explore a state that is the symmetric of one already explored

- Model reformulation
- Addition of constraints (static symmetry breaking)
- During search (dynamic symmetry breaking)
- By dominance detection (dynamic symmetry breaking)

Model reformulation

- Use set variables (inherently unordered)
  - In the Social Golfers example: groups can be represented as sets
  - Only within group symmetry has been removed, but no the groups/weeks/player ones
- Solve a different problem (try to redefine the problem avoiding symmetries)
- Solve the dual problem

Solve a different problem: example

A series is a sequence of twelve tone names (pitch classes) of the chromatic scale, in which each pitch class occurs exactly once. In an all-interval series, also all eleven intervals between the twelve pitches are pairwise distinct.

All-different series

In general words, we are required to find a permutation of the integers $\{0, \ldots, n\}$, such that the differences between adjacent numbers are a permutation of $\{1, \ldots, n\}$.

0 10 1 9 2 8 3 7 4 6 5
10 9 8 7 6 5 4 3 2 1

The problem has many symmetric solutions, e.g. reverse values, “invert” from 10, shifting (according to a pivot), ...

0 10 1 9 2 8 3 7 4 6 5
10 9 8 7 6 5 4 3 2 1

3 7 4 6 5 0 10 1 9 2 8
4 3 2 1 5 10 9 8 7 6

Solve a different problem: example

All-different series: new formulation

Find a permutation of the integers $\{0, \ldots, n\}$ such that:

- the permutation starts with 0, n, 1
- the differences $|x_{i+1} - x_i|$ and $|x_n - x_0|$ are in $\{1, \ldots, n\}$
- exactly one difference occurs twice

This extracts solutions from the original problem with a specific structure

Solve dual problem

- Mainly for value symmetries
- Example: players in golfers
- Consider the dual problem w.r.t. each value v
  - Introduce a set X_v such that

$$i \in X_v \iff y_i = v$$

    (y_i are the original variables)
- Applicable when constraints can be stated easily on the dual problem

Symmetry breaking constraints

- Rule out symmetric solutions by adding further constraints to the original model.
- Assumption: domains are ordered

Lex-leader constraints

Let Σ be the set of all variable symmetry permutations
These symmetry are broken by imposing:

$$[x_1, \ldots, x_n] \preceq_{lex} [x_{\sigma(1)}, \ldots x_{\sigma(n)}], \quad \forall \sigma \in \Sigma$$

Only the lexicographically smallest solution, called lex-leader is preserved

- Distinct integers, $\sigma(1) \neq 1$:
  $[x_1, \ldots, x_n] \preceq_{lex} [x_{\sigma(1)}, \ldots x_{\sigma(n)}] \iff x_1 < x_{\sigma(1)}$
- Disjoint integer sets, $\sigma(1) \neq 1$:
  $[x_1, \ldots, x_n] \preceq_{lex} [x_{\sigma(1)}, \ldots x_{\sigma(n)}] \iff \min(x_1) < \min(x_{\sigma(1)})$
- Arbitrary integers or sets: special propagators

Lex-leader constraints: examples

- n-Queens: $\sigma(i) = n - i + 1$

$$[q_1, \ldots q_n] \preceq_{lex} [q_{\sigma(1)}, \ldots q_{\sigma(n)}] = [q_n, \ldots, q_1]$$

$$\Rightarrow q_1 < q_n$$

- All-Intervals:

$$|x_2 - x_1| > |x_n - x_{n-1}|$$

In Gecode

- Lexicographic constraints between variable arrays. (where the sizes of x and y can be different), If x and y are integer variable arrays

```
rel(home, x, IRT_LE, y);
```

- x is an array of set variables and c is an array of integers

```
precede(home, x, c);
```

  it is enforced that c_k precedes c_{k+1} in x for $0 \leq k < |c| - 1$

Social Golfers

In Gecode

- Using set variables to model the groups avoids introducing symmetry among the players in a group.

```
SetVarArray groups(home,g*w,IntSet::empty,0,g*s-1,s,s);
Matrix<SetVarArray> schedule(groups,g,w);
```

- Within a week, the order of the groups is irrelevant. Static order requiring that all minimal elements of each group are ordered increasingly $\min(groups(g, w)) < \min(group(g + 1, w))$

```
for (int j=0; j<w; j++) {
  IntVarArgs m(g);
  for (int i=0; i<g; i++)
    m[i] = expr(home, min(schedule(i,j)));
  rel(home, m, IRT_LE);
}
```

- similarly, the order of the weeks is irrelevant (remove $\{0\}$ or no effect)

```
IntVarArgs m(w);
for (int j=0; j<w; j++)
  m[j] = expr(home, min(schedule(0,j)-IntSet(0,0)));
rel(home, m, IRT_LE);
```

- the players can be permuted arbitrarily.

```
precede(home, groups, IntArgs::create(g∗s−1, 0)); \\ different from manual
```

$c = (0, \ldots, 14)$: It enforces that for any pair of players c_k and c_{k+1}, $0 \leq k \leq 14$ that c_{k+1} can only appear in a group without c_{k+1} if there is an earlier group where c_k appears without c_{k+1}. Eg, 8 appears in a group without 7 but 7 should appear earlier, hence the constraint is not satisfied.

	group 1			group 2			group 3			group 4			group 5		
week 1	0	1	2	3	4	5	6	9	8	7	10	11	12	13	14
week 2	0	3	6	1	4	7	2	9	12	5	10	13	8	11	14
week 3	0	4	13	1	3	11	2	6	10	5	8	12	9	7	14
week 4	0	5	14	1	10	12	2	3	8	4	9	11	6	7	13
week 5	0	9	10	1	8	13	2	4	14	3	7	12	5	6	11
week 6	0	8	7	1	5	9	2	11	13	3	10	14	4	6	12
week 7	0	11	12	1	6	14	2	5	7	3	9	13	4	8	10

Value symmetries

- Same idea:

$$[x_1, \ldots, x_n] \preceq_{lex} [\sigma(x_1), \ldots \sigma(x_n)], \quad \forall \sigma \in \Sigma$$

- how to implement $\sigma(x_i)$?
- element constraint to implement $\sigma(x_i)$

Example

$\sigma(v) = n - v$

3 7 4 6 5 0 10 1 9 2 8

4 3 2 1 5 10 9 8 7 6

7 3 6 4 5 10 0 9 1 8 2

4 3 2 1 5 10 9 8 7 6

$\sigma = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]$

$[x_0, \ldots, x_{n-1}] \preceq_{lex} [\sigma(x_0), \ldots \sigma(x_{n-1})] \iff x_0 < \sigma(x_0) \iff x_0 < x_1$

Pros and Cons

- Good: for each symmetry, only one solution remains
- Bad:
  may have to ad many constraints
  remaining solution may not be the first one according to branching heuristic!
- Especially bad with dynamic variable selection (like first-fail heuristics)

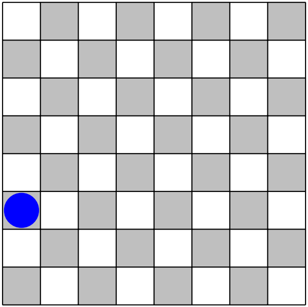
Symmetry Breaking During Search (SBDS)

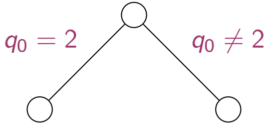
- Add constraints during backtracking to prevent the visit of symmetric search states
- Similar idea to branch-and-bound
- Pros: Works with every type of symmetry
- Cons: Can result in a huge nubmer of constraints to be added, and all symmetries have to be specified explicitly

SBDS Example: n-Queens

Goal: Eliminate r90:

$\{q_i = j\} \in sol(n - \text{Queens}) \iff \{q_j = n - i\} \in sol(n - \text{Queens})$





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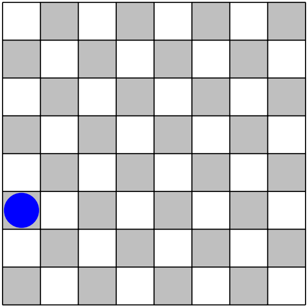

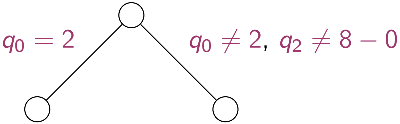

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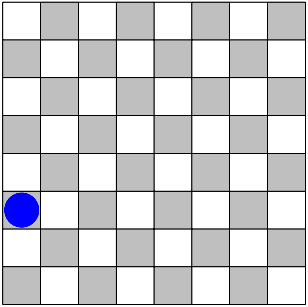

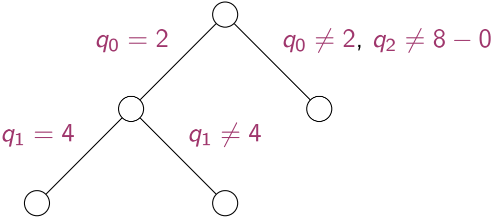

SBDS Example: n-Queens

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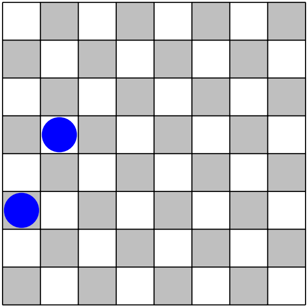

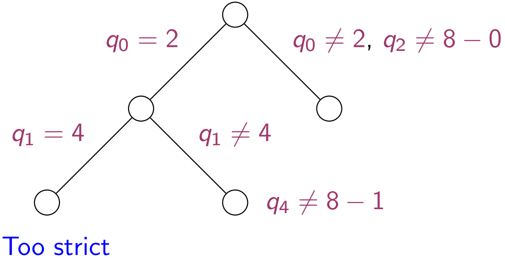

SBDS Example: n-Queens

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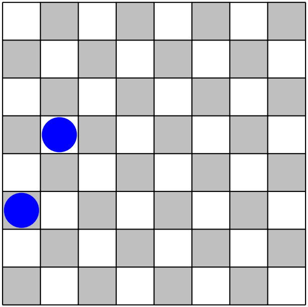
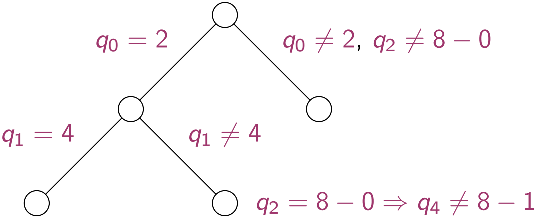




SBDS in group theory perspective

SBDS

For each symmetry g, and a current partial assignment A and choice c, post the constraint:

$$g(A) \Rightarrow \neg g(c)$$

Only interested in different $g(A)$ and $g(c)$

- compute the orbit of the current partial assignment A

Symmetry Breaking by Dominance Detection (SBDD)

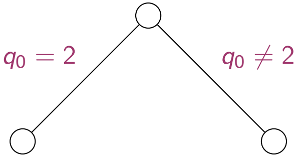
- Do not explore subtrees dominated by a previously visited node
- Multiple definitions of *dominance* are possible
- Pros: No constraints added, very configurable
- Cons: Storage of previous states, checking dominance can be expensive

The idea is similar to *no goods*.
It can be used for propagation.

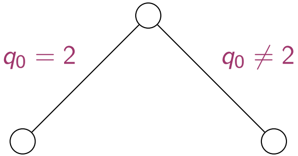
Ingredients

- No-good: A node v is a no-good w.r.t. a node n if there exists an acenstor n_a of n s.t. v is the l;eft hand child of n_a and v is not an ancestor of n.
- Dominance:
  a node n is dominated if there exists a no-good v w.r.t. n and a symmetry g s.t. $(\delta(v))^g \subseteq \mathcal{DE}(n)$
  ($\delta(v)$ set of decisions labelling the path from the root of the tree to the node v)
- Database T of already seen domains

SBDD Example: n-Queens

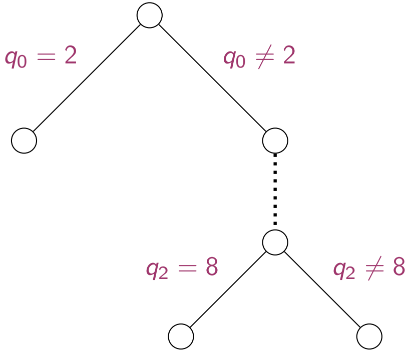


SBDD Example: n-Queens



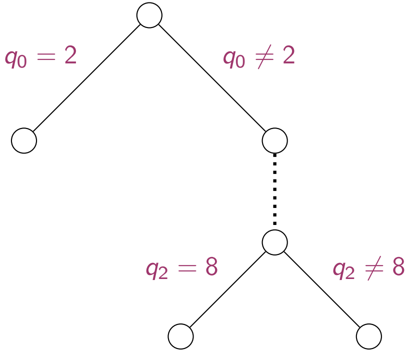
$T = \{\{q_0 = 2\}\}$

SBDD Example: n-Queens



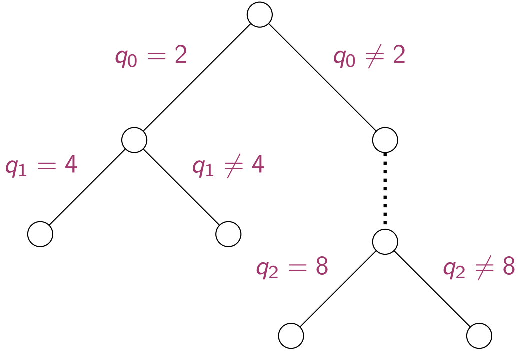
$T = \{\{q_0 = 2\}\}$

SBDD Example: n-Queens

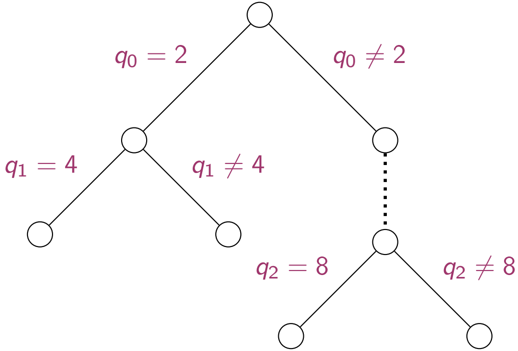


$T = \{\{q_0 = 2\}\}$
Dominated

SBDD Example: n-Queens

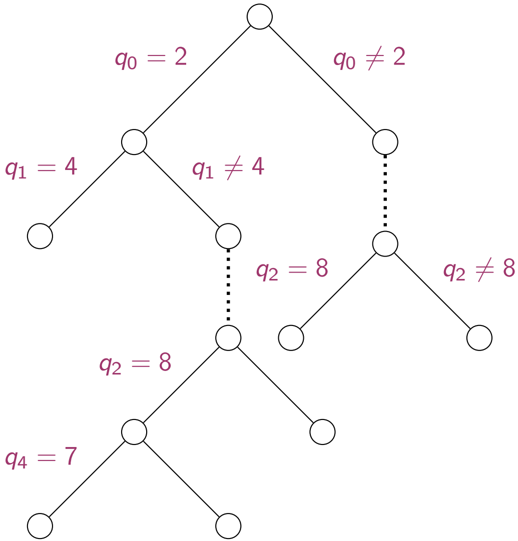


SBDD Example: n-Queens

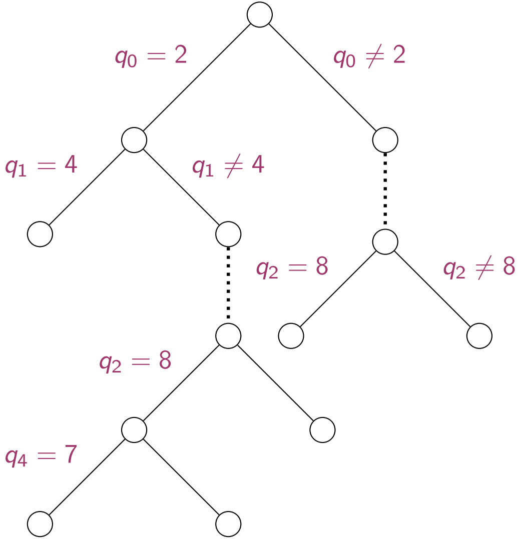


$T = \{\{q_0 = 2, q_1 = 4\}\}$

SBDD Example: n-Queens



SBDD Example: n-Queens



Dominated

SBDD in the group theory perspective

SBDD

A domain d dominates the current node c if c is in the orbit of d

Detection:
function $\Phi : \mathrm{Dom} \times \mathrm{Dom} \mapsto \mathbb{B}$
such that $\Phi(\delta(v), \mathcal{DE}(n)) = \mathit{true}$ iff $\delta(v)$ dominates $\mathcal{DE}(n)$ under some symmetry σ.

Optimization: only keep domains left-adjacent to the path from the root to the current node

Pros and Cons

- Good: No constraints added
- Good: Handles all kinds of symmetry
- Good: V ery configurable (by implementing)
- Bad: Still all symmetries must be encoded
- Bad: Checking dominance at each node may be expensive

References

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Gent I.P., Petrie K.E., and Puget J.F. (2006). **Symmetry in constraint programming**. In *Handbook of Constraint Programming*, edited by F. Rossi, P. van Beek, and T. Walsh, chap. 10, pp. 329–376. Elsevier.