DM826 – Spring 2014

Modeling and Solving Constrained Optimization Problems

Lecture 6

Constraint Propagation Algorithms

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Resume

- Definitions
  (CSP, restrictions, projections, istantiation, local consistency)
- Tigthtenings
- Global consistent (any instantiation local consistent can be extended to a solution) needs exponential time
  $\rightsquigarrow$ local consistency defined by condition Φ of the problem
- Tightenings by constraint propagation: reduction rules + rules iterations
  - reduction rules $\Leftrightarrow$ Φ consistency
  - rules iteration: reach fixed point, that is, closure of all tightenings that are Φ consistent
- Domain-based Φ: (generalized) arc consistency

Outline

Arc Consistency

Arc Consistency rule 1 (propagator):

$$\frac{\langle C; x \in D(x), y \in D(y)\rangle}{\langle C; x \in D'(x), y \in D(y)\rangle}$$

where $D'(x) := \{a \in D(x) | \exists b \in D(y), (a, b) \in C\}$

This can also be written as ($\bowtie$ represents the join):

$$D(x) \leftarrow D(x) \cap \pi_{\{x\}}(\bowtie(C, D(y)))$$

Arc Consistency rule 2 (propagator):

$$\frac{\langle C; x \in D(x), y \in D(y)\rangle}{\langle C; x \in D(x), y \in D'(y)\rangle}$$

where $D'(y) := \{b \in D(y) | \exists a \in D(x), (a, b) \in C\}$

This can also be written as:

$$D(y) \leftarrow D(y) \cap \pi_{\{y\}}(\bowtie(C, D(x)))$$

Generalized Arc Consistency

(Generalized) Arc Consistency rule (propagator):

$$\frac{\langle C; x_1 \in D(x), \dots, x_k \in D(x_k)\rangle}{\langle C; x_1 \in D(x_1), \dots, x_{i-1} \in D(x_{i-1}), x_i \in D'(x_i), x_{i+1} \in D(x_{i+1}), \dots, x_k \in D(x_k)\rangle}$$

where $D'(x_i) := \{a \in D(x_i) | \exists \tau \in C, a = \tau[x_i]\}$

This can also be written as:

$$D(x_i) \leftarrow D(x_i) \cap \pi_{\{x_i\}}(C \cap \pi_{X(C)}(\mathcal{DE}))$$

AC1 – Reduction rule

Revision: making a constraint arc consistent by propagating the domain from a variable to anohter
Corresponds to:

$$D(x) \leftarrow D(x) \cap \pi_{\{x\}}(\bowtie(C, D(y)))$$

for a given variable x and constraint C
Assume normalized network:

REVISE$((x_i), x_j)$
input: a subnetwork defined by two variables $X = \{x_i, x_j\}$, a distinguished variable x_i, domains: D_i and D_j, and constraint R_{ij}
output: D_i, such that, x_i arc-consistent relative to x_j

1. **for** each $a_i \in D_i$
2.   **if** there is no $a_j \in D_j$ such that $(a_i, a_j) \in R_{ij}$
3.   **then** delete a_i from D_i
4.   **endif**
5. **endfor**

Complexity: $O(d^2)$ or $O(rd^r)$
d values, r arity

AC1 – Rules Iteration

Binary case

AC-1($\mathcal{R}$)

input: a network of constraints $\mathcal{R} = (X, D, C)$

output: $\mathcal{R}'$ which is the loosest arc-consistent network equivalent to $\mathcal{R}$

1. **repeat**
2. **for** every pair $\{x_i, x_j\}$ that participates in a constraint
3. Revise$((x_i), x_j)$ (or $D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$)
4. Revise$((x_j), x_i)$ (or $D_j \leftarrow D_j \cap \pi_j(R_{ij} \bowtie D_i)$)
5. **endfor**
6. **until** no domain is changed

- Complexity (Mackworth and Freuder, 1986): $O(end^3)$
  e number of arcs, n variables
  (ed^2 each loop, nd number of loops)
- best-case = $O(ed)$
- Arc-consistency is at least $O(ed^2)$ in the worst case

AC3 (Macworth, 1977)

General case – Arc oriented (coarse-grained)

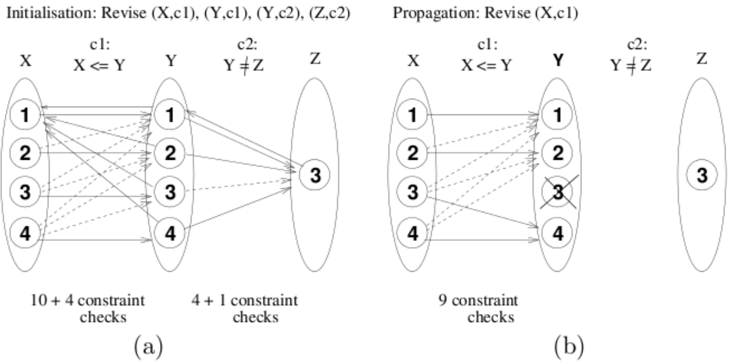
function Revise3(**in** x_i: variable; c: constraint): **Boolean** ;
  begin
    CHANGE $\leftarrow$ **false**;
    foreach $v_i \in D(x_i)$ **do**
      if $\nexists \tau \in c \cap \pi_{X(c)}(D)$ *with* $\tau[x_i] = v_i$ **then**
        remove v_i from $D(x_i)$;
        CHANGE $\leftarrow$ **true**;
    return CHANGE ;
  end

function AC3/GAC3(**in** X: set): **Boolean** ;
  begin
    /* initalisation */;
    $Q \leftarrow \{(x_i, c) \mid c \in C, x_i \in X(c)\}$;
    /* propagation */;
    while $Q \neq \emptyset$ **do**
      select and remove (x_i, c) from Q;
      if Revise(x_i, c) **then**
        if $D(x_i) = \emptyset$ **then** return **false** ;
        else $Q \leftarrow Q \cup \{(x_j, c') \mid c' \in C \wedge c' \neq c \wedge x_i, x_j \in X(c') \wedge j \neq i\}$;
    return **true** ;
  end

$O(er^3 d^{r+1})$ time
$O(er)$ space

AC3
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$



AC4

Binary normalized problems – value oriented (fine grained)

function AC4(**in** X: set)**: Boolean** ;
  begin
    /* initialization */;
    $Q \leftarrow \emptyset$; $S[x_j, v_j] = 0, \forall v_j \in D(x_j), \forall x_j \in X$;
    foreach $x_i \in X, c_{ij} \in C, v_i \in D(x_i)$ **do**
      initialize counter$[x_i, v_i, x_j]$ to $|\{v_j \in D(x_j) \mid (v_i, v_j) \in c_{ij}\}|$;
      if *counter*$[x_i, v_i, x_j] = 0$ **then** remove v_i from $D(x_i)$ and add (x_i, v_i) to Q;
      add (x_i, v_i) to each $S[x_j, v_j]$ s.t. $(v_i, v_j) \in c_{ij}$;
      if $D(x_i) = \emptyset$ **then** return **false** ;
    /* propagation */;
    while $Q \neq \emptyset$ **do**
      select and remove (x_j, v_j) from Q;
      foreach $(x_i, v_i) \in S[x_j, v_j]$ **do**
        if $v_i \in D(x_i)$ **then**
          counter$[x_i, v_i, x_j]$ = counter$[x_i, v_i, x_j] - 1$;
          if *counter*$[x_i, v_i, x_j] = 0$ **then**
            remove v_i from $D(x_i)$; add (x_i, v_i) to Q;
            if $D(x_i) = \emptyset$ **then** return **false** ;
    return **true** ;
  end

$O(ed^2)$ time
$O(erd^r)$ time for GAC

AC4
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$

$$\begin{array}{lll}
\texttt{counter}[x,1,y] = 4 & \texttt{counter}[y,1,x] = 1 & \texttt{counter}[y,1,z] = 1 \\
\texttt{counter}[x,2,y] = 3 & \texttt{counter}[y,2,x] = 2 & \texttt{counter}[y,2,z] = 1 \\
\texttt{counter}[x,3,y] = 2 & \texttt{counter}[y,3,x] = 3 & \texttt{counter}[y,3,z] = 0 \\
\texttt{counter}[x,4,y] = 1 & \texttt{counter}[y,4,x] = 4 & \texttt{counter}[y,4,z] = 1 \\
 & & \texttt{counter}[z,3,y] = 3
\end{array}$$

$$\begin{array}{lr}
S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} & S[y,1] = \{(x,1),(z,3)\} \\
S[x,2] = \{(y,2),(y,3),(y,4)\} & S[y,2] = \{(x,1),(x,2),(z,3)\} \\
S[x,3] = \{(y,3),(y,4)\} & S[y,3] = \{(x,1),(x,2),(x,3)\} \\
S[x,4] = \{(y,4)\} & S[y,4] = \{(x,1),(x,2),(x,3),(x,4),(z,3)\} \\
 & S[z,3] = \{(y,1),(y,2),(y,4)\}
\end{array}$$

AC6

Binary normalized problems

function `AC6`(**in** X: set)**: Boolean** ;
  begin
    /* initialization */;
    $Q \leftarrow \emptyset$; $S[x_j, v_j] = 0, \forall v_j \in D(x_j), \forall x_j \in X$;
    foreach $x_i \in X, c_{ij} \in C, v_i \in D(x_i)$ **do**
      $v_j \leftarrow$ smallest value in $D(x_j)$ s.t. $(v_i, v_j) \in c_{ij}$;
      if v_j *exists* **then** add (x_i, v_i) to $S[x_j, v_j]$;
      else remove v_i from $D(x_i)$ and add (x_i, v_i) to Q;
      if $D(x_i) = \emptyset$ **then** return **false** ;
    /* propagation */;
    while $Q \neq \emptyset$ **do**
      select and remove (x_j, v_j) from Q;
      foreach $(x_i, v_i) \in S[x_j, v_j]$ **do**
        if $v_i \in D(x_i)$ **then**
          $v'_j \leftarrow$ smallest value in $D(x_j)$ greater than v_j s.t. $(v_i, v_j) \in c_{ij}$;
          if v'_j *exists* **then** add (x_i, v_i) to $S[x_j, v'_j]$;
          else
            remove v_i from $D(x_i)$; add (x_i, v_i) to Q;
            if $D(x_i) = \emptyset$ **then** return **false** ;
    return **true** ;
  end

$O(ed^2)$ time
$O(ed)$ space

AC6
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$

$$\begin{aligned} S[x,1] &= \{(y,1),(y,2),(y,3),(y,4)\} \\ S[x,2] &= \{\} \\ S[x,3] &= \{\} \\ S[x,4] &= \{\} \end{aligned}$$

$$\begin{aligned} S[y,1] &= \{(x,1),(z,3)\} \\ S[y,2] &= \{(x,2)\} \\ S[y,3] &= \{(x,3)\} \\ S[y,4] &= \{(x,4)\} \\ S[z,3] &= \{(y,1),(y,2),(y,4)\} \end{aligned}$$

Reverse2001

Binary case

function Revise2001(**in** x_i: variable; c_{ij}: constraint)**: Boolean** ;
  begin
    CHANGE ← **false**;
    foreach $v_i \in D(x_i)$ *s.t.* $Last(x_i, v_i, x_j) \notin D(x_j)$ **do**
      $v_j \leftarrow$ smallest value in $D(x_j)$ greater than Last(x_i, v_i, x_j) s.t. $(v_i, v_j) \in c_{ij}$;
      if v_j *exists* **then** Last$(x_i, v_i, x_j) \leftarrow v_j$;
      else
        remove v_i from $D(x_i)$;
        CHANGE ← **true**;
    return CHANGE ;
  end

function AC3/GAC3(**in** X: set)**: Boolean** ;
  begin
    /* initalisation */;
    $Q \leftarrow \{(x_i, c) \mid c \in C, x_i \in X(c)\}$;
    /* propagation */;
    while $Q \neq \emptyset$ **do**
      select and remove (x_i, c) from Q;
      if *Revise*(x_i, c) **then**
        if $D(x_i) = \emptyset$ **then** return **false** ;
        else $Q \leftarrow Q \cup \{(x_j, c') \mid c' \in C \wedge c' \neq c \wedge x_i, x_j \in X(c') \wedge j \neq i\}$;
    return **true** ;
  end

$O(ed^2)$ time
$O(ed)$ space

Reverse2001
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$

$$\begin{array}{lll}
\texttt{Last}[x, 1, y] = 1 & \texttt{Last}[y, 1, x] = 1 & \texttt{Last}[y, 1, z] = 3 \\
\texttt{Last}[x, 2, y] = 2 & \texttt{Last}[y, 2, x] = 1 & \texttt{Last}[y, 2, z] = 3 \\
\texttt{Last}[x, 3, y] = 3 & \texttt{Last}[y, 3, x] = 1 & \texttt{Last}[y, 3, z] = nil \\
\texttt{Last}[x, 4, y] = 4 & \texttt{Last}[y, 4, x] = 1 & \texttt{Last}[y, 4, z] = 3 \\
 & & \texttt{Last}[z, 3, y] = 1
\end{array}$$

Limitation of Arc Consistency

Example

$$\langle x < y, y < z, z < x; x, y, z \in \{1..100000\}\rangle$$

is inconsistent.

Proof: Apply revise to $(x, x < y)$

$$\langle x < y, y < z, z < x; x \in \{1..99999\}, y, z \in \{1..100000\}\rangle,$$

ecc. we end in a fail.

- Disadvantage: large number of steps.
  Run time depends on the size of the domains!
- Note: we could prove fail by transitivity of $<$.
  $\leadsto$ Path consitency involves two constraints together