DM826 – Spring 2014

Modeling and Solving Constrained Optimization Problems

Lecture 8

Constraint Propagation

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[*Partly based on slides by Christian Schulte, KTH Royal Institute of Technology*]

Outline

1. Generic Rules Iteration

2. Systems

Algorithms for constraint propagation:

- scheduling steps of atomic reduction
- termination criterion: local consistency

- How to schedule the application of reduction rules to guarantee termination?
- How to avoid (at low cost) the application of redundant rules?
- Have all derivations the same result?
- How can we characterize it?

Propagators

- Given $\mathcal{P}$ a reduction rule is a function f from $\mathcal{S}_{\mathcal{P}}$ to $\mathcal{S}_{\mathcal{P}}$ for all $\mathcal{P}' \in \mathcal{S}_{\mathcal{P}}$, $f(\mathcal{P}') \in \mathcal{S}_{\mathcal{P}}$.
  (most cases take care of one a single variable and a single constraints):
- Given $\mathcal{C}$ in $\mathcal{P}$ a propagator f for $\mathcal{C}$ is a reduction rule from $\mathcal{S}_{\mathcal{P}}$ to $\mathcal{S}_{\mathcal{P}}$ that tightens only domains independently of the constraints other than C.
- A propagator f can be seen as a function: $f : S \to S$
- A propagator f is correct for C iff it does not remove any assignment for C: $\{a \in D\} \cap C = \{a \in f(D)\} \cap C$

Systems consider set of propagators to implement a constraint
(However global constraints have a single propagator.)

Example

$C \equiv x_1 \leq x_2 + 1$

$$p(D)(x_1) = \{n \in D(X_1) | n \leq \max_D\{x_2\} + 1\}$$

$\text{input}(p) = x_2$, $\text{output}(p) = x_1$

Propagators

- Properties of propagators:
  Given $\mathcal{P}$, f can be:
  - contracting (or decreasing): $f(\mathcal{P}) \leq \mathcal{P}$
  - monotonic if $\mathcal{P}_1 \leq \mathcal{P}_2 \Rightarrow f(\mathcal{P}_1) \leq f(\mathcal{P}_2)$
  - idempotent if $f(f(\mathcal{P})) = f(\mathcal{P})$ (strong if for all $\mathcal{P} \in \mathcal{SP}$, weak if for some $\mathcal{P} \in \mathcal{SP}$)
  - commuting if $fg(\mathcal{P}) = gf(\mathcal{P})$
  - subsumed (or entailed) by $\mathcal{P}$ iff $\forall \mathcal{P}_1 \leq \mathcal{P} : f(\mathcal{P}_1) = \mathcal{P}_1$
    Eg:

    $$p(D)(x) = D(x) \cap \{1, 2, 3\}$$

    implementing the domain constraint $x \in \{1, 2, 3\}$. After p has been executed once, there is no point to execute p again as for all D'
    $D' \leq p(D) \implies p(D') = D'$
    (particular case when all variables are instantiated)

- Iteration: Let $\mathcal{P} = \langle X, \mathcal{DE}, \mathcal{C} \rangle$ and $F = \{f_1, \ldots, f_k\}$ a finite set of propagators on $\mathcal{S}_{\mathcal{P}}$. An iteration of F on $\mathcal{P}$ is a sequence $\langle \mathcal{P}_0, \mathcal{P}_1, \ldots \rangle$ of elements of $\mathcal{S}_{\mathcal{P}}$ defined by

$$\mathcal{P}_0 = \mathcal{P}$$

$$\mathcal{P}_j = f_{n_j}(\mathcal{P}_{j-1})$$

  where $j > 0$ and $n_j \in [1, \ldots, k]$.
- $\mathcal{P}$ is stable for F iff $\forall f \in F, f(\mathcal{P}) = \mathcal{P}$
- there may be several stable $\mathcal{P}$ but if F are monotonic then unique
- Let $\mathcal{P} = \langle X, \mathcal{DE}, \mathcal{C} \rangle$ and $F = \{f_1, \ldots, f_k\}$. If $\langle \mathcal{P}_0, \mathcal{P}_1, \ldots \rangle$is infinite iteration of F where each $f \in F$ is activated infinitely often then there exists $j \geq 0$ such that $\mathcal{P}_j$ is stable for F ($\equiv j$ is finite!)
- If $\mathcal{P}$ is stable for F then it is its weakest simultaneous fixed point (greatest mutual fixed point of all propagators).
  A strongest simultaneous fixed point would be a solution (hence not unique) which would violate solution preservation

Iteration of Reduction Rules

procedure *Generic-Iteration*(N, F);
    $G \leftarrow F$;
    while $G \neq \emptyset$ **do**
        select and remove g from G;
        if $N \neq g(N)$ **then**
            $update(G)$;
            $N \leftarrow g(N)$;

/* $update(G)$ adds to G at least all functions f in $F \setminus G$ for which $g(N) \neq f(g(N))$ */

If the propagator is contracting then Generic-Iteration terminates.
If propagator is monotonic then the final result does not change with the order in which propagators are applied.

If propagators in addition to monotonic are also idempotent and commutative then:

procedure *Direct-Iteration*(N, F);
    $G \leftarrow F$;
    while $G \neq \emptyset$ **do**
        select and remove g from G;
        $N \leftarrow g(N)$;

Iteration of Reduction Rules

Example

Recall for arc consistency:
Arc Consistency rule 1 (propagator):

$$\frac{\langle C; x \in D(x), y \in D(y)\rangle}{\langle C; x \in D'(x), y \in D(y)\rangle}$$

where $D'(x) := \{a \in D(x) | \exists b \in D(y), (a, b) \in C\}$

This can also be written as ($\bowtie$ represents the join):

$$D(x) \leftarrow D(x) \cap \pi_{\{x\}}(\bowtie(C, D(y)))$$

$$\forall N_1 = (X, D_1, C) \in \mathcal{P}_{ND}, \forall x_i \in X, \forall c_j \in C,\ f_{i,j}(N_1) = (X, D_1', C) \ \text{ with}$$

$$D_1'(x_i) = \pi_{\{x_i\}}(c_j \cap \pi_{X(c_j)}(D_1)) \ \text{ and } \ D_1'(x_k) = D_1(x_k), \forall k \neq i.$$

Set of propagators $F_{AC} = \{f_{ij} \mid x_i \in X, c_j \in \mathcal{C}\}$ all monotonic. $\Rightarrow$ terminates in arc consistency closure, which is fixed point for F_{AC}.

Improvements

Generic iteration is an example of propagator engine

```
propagate(P_f, P_n, D)
1:  N ← P_n
2:  P ← P_f ∪ P_n
3:  while N ≠ ∅ do
4:     p ← select(N)
5:     N ← N − {p}
6:     D′ ← p(D)
7:     M ← {x ∈ 𝒱 | D(x) ≠ D′(x)}
8:     N ← N ∪ {p′ ∈ P | input(p′) ∩ M ≠ ∅}
11:    D ← D′
12: return D
```

P_f is set of propagators at fixed point (idempotent or subsumed)

Scheduling p: adding a propagator to the set N (not known to be at fixed point). Yet undefined how a propagator is chosen from N

Note: search can be seen as doing incremental propagation

Improvements

Generic iteration is an example of propagator engine
What makes it naive?

- Termination relies on strict contraction
- We always have to check all propagators for one that can strictly contract

Ideas:

- Maintain propagators which are known to be at fixpoint
- Look at the variables that propagators actually compute with
  Dependency-directed propagation

Fixpoint knowledge avoids useless execution (idempotence, subsumption)
knowledge provided by propagator

Improvements: Events

Most solvers impleemnt arithemitc-oriented propagators
$\rightsquigarrow$ a reduction of a domain of a variable has different implications depending on the type of reduction

Four types of Events:

- Any or RemValue: when a value v is removed from $D(x_i)$
- Min or IncMin: when the minimum value of $D(x_i)$ increases
- Max or DecMax: when the maximum value of $D(x_i)$ decreases
- Fix or Instantiate: when $D(x_i)$ becomes a singleton

AC3 like

Modified AC3 to handle parameter `Mtype` (modification type)

function `Constraint-Propag`(**in** X: set): **Boolean** ;
  begin
    foreach $c \in C$ **do** perform `init-propag` on c and update Q with relevant events;
    while $Q \neq \emptyset$ **do**
      select and remove $(x_i, c, x_j, Mtype)$ from Q;
      if *Revise*$(x_i, c, (x_j, Mtype), Changes)$ **then**
        if $D(x_i) = \emptyset$ **then** return **false** ;
        foreach $c' \in \Gamma^C(x_i), Mtype \in Changes$ **do**
          foreach $x_j \in X(c'), j \neq i$ **do** $Q \leftarrow Q \cup \{(x_j, c', x_i, Mtype)\}$;
    return **true** ;
  end
  /* $\Gamma^C(x_i)$ is the set of constraints with x_i in their scheme */

The presence of $(x_j, c, x_i, \texttt{Mtype})$ in Q means that x_j should be revised on c because of an M type change in $D(x_i)$.

Process constraint propagation differently according to the type of event

function revise(**inout** x_i; **in** $c \equiv x_{k_1} \leq x_{k_2}$; **in** $(x_j, Mtype)$; **out** $Changes$):
Boolean ;
$Changes \leftarrow \emptyset$;
switch $Mtype$ **do**
  case *RemValue*
    nothing;
  case *IncMin*
    if $j = k_1$ **then** remove all $v < min_D(x_j)$ from $D(x_i)$;
  case *DecMax*
    if $j = k_2$ **then** remove all $v > max_D(x_j)$ from $D(x_i)$;
  case *Instantiate*
    if $j = k_1$ **then** remove all $v < min_D(x_j)$ from $D(x_i)$;
    else remove all $v > max_D(x_j)$ from $D(x_i)$;
$Changes \leftarrow$ the types of changes performed on $D(x_i)$;

Also: for a certain constraint it can be that a given event cannot alter the other variables of the constraint. Hence it makes sense to:
6: **foreach** $c' \in \Gamma^c_{\texttt{Mtype}}(x_i), \texttt{Mtype} \in \mathit{Changes}$ **do** ...
Example. Let $c \equiv x_1 \leq x_2$. The only events that require propagation are IncMin and Instantiate on x_1 , and DecMax and Instantiate on x_2.

select and remove $(x_i, c, x_j, Mtype, \Delta_j)$ from Q;
if Revise$(x_i, c, (x_j, Mtype, \Delta_j), Changes, \Delta_i)$ **then**
  if $D(x_i) = \emptyset$ **then** return **false**;
  foreach $c' \in \Gamma^C_{Mtype}(x_i), Mtype \in Changes$ **do**
    foreach $x_j \in X(c'), j \neq i$ **do** $Q \leftarrow Q \cup \{(x_j, c', x_i, Mtype, \Delta_i)\}$

---

function revise(**inout** x_i; **in** $c \equiv x_{k_1} = x_{k_2} + m$; **in** $(x_j, Mtype, \Delta_j)$;
out $Changes$; **out** Δ_i)**: Boolean** ;

$Changes \leftarrow \emptyset$;
switch $Mtype$ **do**
  case *RemValue*
    if $j = k_1$ **then** **foreach** $v \in \Delta_j$ **do** remove $(v - m)$ from $D(x_i)$;
    else **foreach** $v \in \Delta_j$ **do** remove $(v + m)$ from $D(x_i)$;
  case *IncMin*
    if $j = k_1$ **then** remove all $v < min_D(x_j) - m$ from $D(x_i)$;
    else remove all $v < min_D(x_j) + m$ from $D(x_i)$;
  case *DecMax*
    if $j = k_1$ **then** remove all $v > max_D(x_j) - m$ from $D(x_i)$;
    else remove all $v > max_D(x_j) + m$ from $D(x_i)$;
  case *Instantiate*
    if $j = k_1$ **then** assign $min_D(x_j) - m$ to x_i;
    else assign $min_D(x_j) + m$ to x_i;
$Changes \leftarrow$ the types of changes performed;
$\Delta_i \leftarrow$ all values removed from $D(x_i)$;

More Optimization

Priorities
Choose propagator

- according to cost: cheapest first
- according to expected impact
- general (queue): last-in last-out (starvation avoided), first-in first-out

Propagator Rewriting

Another observation:
propagator for

$$\max(x, y) = z$$

and values for x are smaller than for y
Replace by propagator for $y = z$

Outline

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2. Systems

Architecture

- Detecting failure and entailment
- Domains: single data structure continously updated.
  constraint store $\equiv$ domain extension $\mathcal{DE}$
- State restoration
- Finding dependent propagators (compute events and find propagators)
- Variables for proagators

Propagation Services

- Events
- Selecting next propagator

Variable Domains

- Domain representation
  range sequence: $s = \{[n_1, m_1], \ldots, [n_k, m_k]\}$ (singly/doubly linked lists)
  bit vector
- Value operations
  `x.getmin()`, `x.getmax()`, `x.hasval()`, `x.adjmin(n)`, `x.adjmax(n)`, `x.excval(n)`
- Iterators:

```
for (IntVarValues i(x); i(); ++i)
  std::cout << i.val() << ' ';

for (IntVarRanges i(x); i(); ++i)
  std::cout << i.min() << ".." << i.max() << ' ';
```

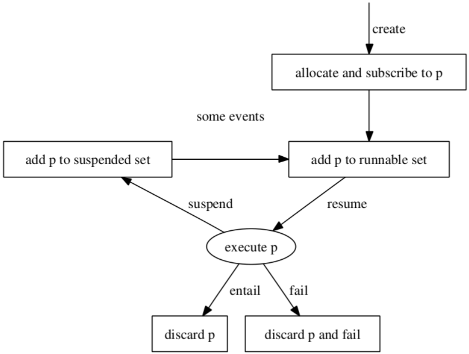
- Domain operations
- Subscriptions
  - list $E_i.p_i$ pair event propagator that require execution
  - a list for each event and one for each propagator
  - array of propagators partitioned by events

Operations	Range sequence	Bitvector
x.getmin()	$O(1)$	$O(1)$
x.getmax()	$O(1)$	$O(1)$
x.hasval(n)	$O(r)$	$O(1)$
x.adjmin(n)	$O(r)$	$O(1)$
x.adjmax(n)	$O(r)$	$O(1)$
x.excval(n)	$O(r)$	$O(v)$
i.done()	$O(1)$	$O(v)$
i.value()	$O(1)$	$O(1)$
i.next()	$O(1)$	$O(v)$

Propagators

Piece of software with some private state that implements a constraint C over some variables or *parameters*

The algorithm implemented is called filtering algorithm. It uses value and domain operations and raises events that cause scheduling of other propagators
Life cycle



- Idempotency: it may be costly and difficult to guarantee. Some propagators return a state:
  - fixpoint (weak idempotent),
  - no fixpoint (we do not know),
  - subsumed (entailed),
  - failure.

References

Bessiere C. (2006). **Constraint propagation**. In *Handbook of Constraint Programming*, edited by F. Rossi, P. van Beek, and T. Walsh, chap. 3. Elsevier. Also as Technical Report LIRMM 06020, March 2006.

Schulte C. and Carlsson M. (2006). **Finite domain constraint programming systems**. In *Handbook of Constraint Programming*, edited by F. Rossi, P. van Beek, and T. Walsh. Elsevier.