DM545

Linear and Integer Programming

Lecture 13
Cutting Planes and Branch and Bound

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Outline

1. Cutting Plane Algorithms

2. Branch and Bound

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1. Cutting Plane Algorithms

2. Branch and Bound

Valid Inequalities

- IP: $z = \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\}, X = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_+\}$
- Proposition: $\text{conv}(X) = \{\mathbf{x} : \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}, \mathbf{x} \geq 0\}$ is a polyhedron
- LP: $z = \max\{\mathbf{c}^T\mathbf{x} : \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}, \mathbf{x} \geq \mathbf{0}\}$ would be the best formulation
- Key idea: try to approximate the best formulation.

Definition (Valid inequalities)

$\mathbf{ax} \leq \mathbf{b}$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\mathbf{ax} \leq \mathbf{b}\ \forall \mathbf{x} \in X$

Which are useful inequalities? and how can we find them?
How can we use them?

Example: Pre-processing

- $X = \{(x, y) : x \leq 999y; 0 \leq x \leq 5, y \in \mathbb{B}^1\}$

$$x \leq 5y$$

- $X = \{x \in \mathbb{Z}^n_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6 + \frac{6}{11}$$
$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

- Capacitated facility location:

$$\sum_{i \in M} x_{ij} \leq b_j y_j \quad \forall j \in N \qquad\qquad x_{ij} \leq b_j y_j$$
$$\sum_{j \in N} x_{ij} = a_i \quad \forall i \in M \qquad\qquad x_{ij} \leq a_i$$
$$x_{ij} \geq 0, y_j \in B^n \qquad\qquad x_{ij} \leq \min\{a_i, b_j\} y_j$$

Chvátal-Gomory cuts

- $X \in P \cap \mathbb{Z}_+^n, \quad P = \{\mathbf{x} \in \mathbb{R}_+^n : A\mathbf{x} \leq \mathbf{b}\}, \quad A \in \mathbb{R}^{m \times n}$
- $\mathbf{u} \in \mathbb{R}_+^m$, $\{\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_n\}$ columns of A

CG procedure to construct valid inequalities

1) $\sum_{j=1}^{n} \mathbf{u}\mathbf{a}_j x_j \leq \mathbf{u}\mathbf{b}$ valid: $\mathbf{u} \geq \mathbf{0}$

2) $\sum_{j=1}^{n} \lfloor \mathbf{u}\mathbf{a}_j \rfloor x_j \leq \mathbf{u}\mathbf{b}$ valid: $\mathbf{x} \geq \mathbf{0}$ and $\sum \lfloor \mathbf{u}\mathbf{a}_j \rfloor x_j \leq \sum \mathbf{u}\mathbf{a}_j x_j$

3) $\sum_{j=1}^{n} \lfloor \mathbf{u}\mathbf{a}_j \rfloor x_j \leq \lfloor \mathbf{u}\mathbf{b} \rfloor$ valid for X since $\mathbf{x} \in \mathbb{Z}^n$

Theorem

by applying this CG procedure a finite number of times every valid inequality for X can be obtained

Cutting Plane Algorithms

- $X \in P \cap \mathbb{Z}^n_+$
- a family of valid inequalities $\mathcal{F} : \mathbf{a}^T\mathbf{x} \leq b, (\mathbf{a}, b) \in \mathcal{F}$ for X
- we do not find them all a priori, only interested in those close to optimum

Cutting Plane Algorithm

Init.: $t = 0, P^0 = P$

Iter. t: Solve $\bar{z}^t = \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P^t\}$
let $\mathbf{x}^t$ be an optimal solution
if $\mathbf{x}^t \in \mathbb{Z}^n$ stop, $\mathbf{x}^t$ is opt to the IP
if $\mathbf{x}^t \notin \mathbb{Z}^n$ solve separation problem for $\mathbf{x}^t$ and $\mathcal{F}$
if $(\mathbf{a}^t, b^t)$ is found with $\mathbf{a}^t\mathbf{x}^t > b^t$ that cuts off x^t

$$P^{t+1} = P \cap \{\mathbf{x} : \mathbf{a}^i\mathbf{x} \leq b^i, i = 1, \ldots, t\}$$

else stop (P^t is in any case an improved formulation)

Gomory's fractional cutting plane algorithm

Cutting plane algorithm + Chvátal-Gomory cuts

- $\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{Z}^n\}$
- Solve LPR to optimality

$$\left[\begin{array}{c|c|c|c} I & \bar{A}_N = A_B^{-1}A_N & 0 & \bar{b} \\ \hline \bar{c}_B & \bar{c}_N(\leq 0) & 1 & -\bar{d} \end{array}\right]$$

$$x_u = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j, \quad u \in B$$
$$z = \bar{d} + \sum_{j \in N} \bar{c}_j x_j$$

- If basic optimal solution to LPR is not integer then $\exists$ some row u: $\bar{b}_u \notin \mathbb{Z}^1$.
  The Chvatál-Gomory cut applied to this row is:

$$x_{B_u} + \sum_{j \in N} \lfloor \bar{a}_{uj} \rfloor x_j \leq \lfloor \bar{b}_u \rfloor$$

(B_u is the index in the basis B corresponding to the row u) (cntd)

- Eliminating $x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j$ in the CG cut we obtain:

$$\sum_{j \in N} (\underbrace{\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor}_{0 \leq f_{uj} < 1}) x_j \geq \underbrace{\bar{b}_u - \lfloor \bar{b}_u \rfloor}_{0 < f_u < 1}$$

$$\sum_{j \in N} f_{uj} x_j \geq f_u$$

  $f_u > 0$ or else u would not be row of fractional solution. It implies that x^* in which $x^*_N = 0$ is cut out!

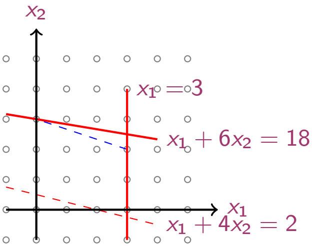
- Moreover: when x is integer, since all coefficient in the CG cut are integer the slack variable of the cut is also integer:

$$s = -f_u + \sum_{j \in N} f_{uj} x_j$$

(theoretically it terminates after a finite number of iterations, but in practice not successful.)

Example

$$\begin{aligned} \max\ x_1 + 4x_2 & \\ x_1 + 6x_2 &\leq 18 \\ x_1 \qquad &\leq 3 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\text{integer} \end{aligned}$$



	x1	x2	x3	x4	-z	b
	1	6	1	0	0	18
	1	0	0	1	0	3
	1	4	0	0	1	0

	x1	x2	x3	x4	-z	b
	0	1	1/6	-1/6	0	15/6
	1	0	0	1	0	3
	0	0	-2/3	-1/3	1	-13

$x_2 = 5/2, x_1 = 3$

Optimum, not integer

- We take the first row:
  `|     |  0  |  1  | 1/6  | -1/6 |  0  | 15/6 |`

- CG cut $\sum_{j \in N} f_{uj} x_j \geq f_u \rightsquigarrow \frac{1}{6} x_3 + \frac{5}{6} x_4 \geq \frac{1}{2}$

- Let's see that it leaves out x^*: from the CG proof:

$$\begin{array}{l} 1/6\,(x_1 + 6x_2 \leq 18) \\ 5/6\,(x_1 \qquad\quad \leq 3) \\ \hline \qquad x_1 + x_2 \leq 3 + 5/2 = 5.5 \end{array}$$

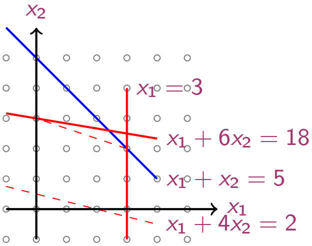
  since x_1, x_2 are integer $x_1 + x_2 \leq 5$

- Let's see how it looks in the space of the original variables: from the first tableau:

$$x_3 = 18 - 6x_2 - x_1$$
$$x_4 = 3 - x_1$$

$$\frac{1}{6}(18 - 6x_2 - x_1) + \frac{5}{6}(3 - x_1) \geq \frac{1}{2} \qquad \rightsquigarrow \qquad x_1 + x_2 \leq 5$$

- Graphically:



- Let's continue:

	x1	x2	x3	x4	x5	-z	b
	0	0	-1/6	-5/6	1	0	-1/2
	0	1	1/6	-1/6	0	0	5/2
	1	0	0	1	0	0	3
	0	0	-2/3	-1/3	0	1	-13

We need to apply dual-simplex (will always be the case, why?)

ratio rule: $\min \left| \frac{c_j}{a_{ij}} \right|$

- After the dual simplex iteration:

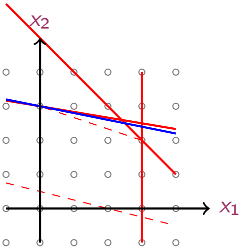
	x1	x2	x3	x4	x5	-z	b
	0	0	1/5	1	-6/5	0	3/5
	0	1	1/5	0	-1/5	0	13/5
	1	0	-1/5	0	6/5	0	12/5
	0	0	-3/5	0	-2/5	1	-64/5

We can choose any of the three rows.

Let's take the third: CG cut:
$\frac{4}{5}x_3 + \frac{1}{5}x_5 \geq \frac{2}{5}$

- In the space of the original variables:

$$4(18 - x_1 - 6x_2) + (5 - x_1 - x_2) \geq 2$$
$$x_1 + 5x_2 \leq 15$$



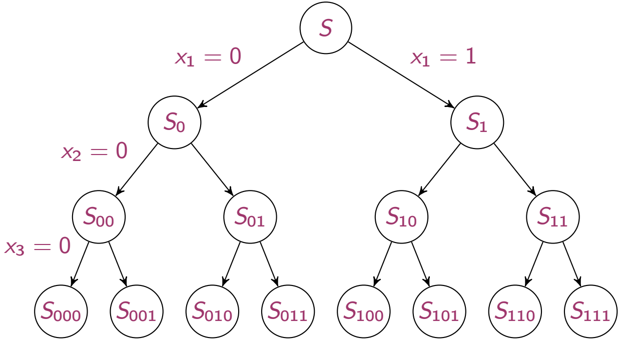
- ...

Outline

Branch and Bound

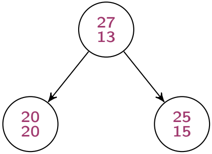
- Consider the problem $z = \max\{c^T x : x \in S\}$
- Divide and conquer: let $S = S_1 \cup \ldots \cup S_k$ be a decomposition of S into smaller sets, and let $z^k = \max\{c^T x : x \in S_k\}$ for $k = 1, \ldots, K$. Then $z = \max_k z^k$

For instance if $S \subseteq \{0,1\}^3$ the enumeration tree is:



Bounding

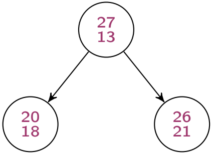
- Let $\overline{z}^k$ be an upper bound on z^k
- Let $\underline{z}^k$ be an lower bound on z^k
- $(\underline{z}^k \leq z^k \leq \overline{z}^k)$
- $\overline{z} = \max_k \overline{z}^k$ is an upper bound on z
- $\underline{z} = \max_k \underline{z}^k$ is a lower bound on z



$\bar{z} = 25$

$\underline{z} = 20$

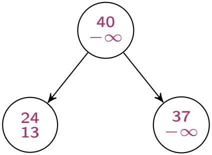
pruned by optimality



$\bar{z} = 26$

$\underline{z} = 21$

pruned by bounding



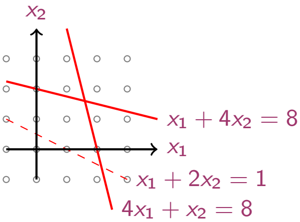
$\bar{z} = 37$

$\underline{z} = 13$

nothing to prune

Example

$$
\begin{aligned}
\max \quad & x_1 + 2x_2 \\
& x_1 + 4x_2 \leq 8 \\
& 4x_1 + x_2 \leq 8 \\
& x_1, x_2 \geq 0, \text{integer}
\end{aligned}
$$



- Solve LP

	x1	x2	x3	x4	-z	b
	1	4	1	0	0	8
	4	1	0	1	0	8
	1	2	0	0	1	0

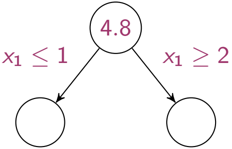
	x1	x2	x3	x4	-z	b
I'=I-II'	0	15/4	1	-1/4	0	6
II'=1/4II	1	1/4	0	1/4	0	2
III'=III-II'	0	7/4	0	-1/4	0	-2

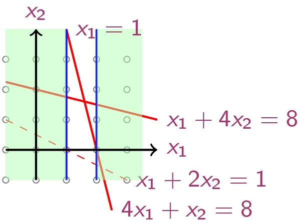
- continuing

	x1	x2	x3	x4	-z	b
I'=4/15I	0	1	4/15	-1/15	0	24/15
II'=II-1/4I'	1	0	-1/15	4/15	0	24/15
III'=III-7/4I'	0	0	-7/15	-3/5	1	-2-14/5

$x_2 = 1 + 3/5 = 1.6$
$x_1 = 8/5$
The optimal solution will not be more than $2 + 14/5 = 4.8$

- Both variables are fractional, we pick one of the two:





- Let's consider first the left branch:

	x1	x2	x3	x4	x5	-z	b
	1	0	0	0	1	0	1
	0	1	4/15	-1/15	0	0	24/15
	1	0	-1/15	4/15	0	0	24/15
	0	0	-7/15	-3/5	0	1	-24/5

	x1	x2	x3	x4	x5	b	-z
I'=I-III	0	0	1/15	-4/15	1	0	-9/15
	0	1	4/15	-1/15	0	0	24/15
	1	0	-1/15	4/15	0	0	24/15
	0	0	-7/15	-3/5	0	1	-24/5

	x1	x2	x3	x4	x5	b	-z
I'=-15/4I	0	0	-1/4	1	-15/4	0	9/4
II'=II-1/4I	0	1	15/60	0	-1/4	0	7/4
III'=III+I	1	0	0	0	1	0	1
	0	0	-37/60	0	-9/4	1	-90/20

always a b term negative after branching:

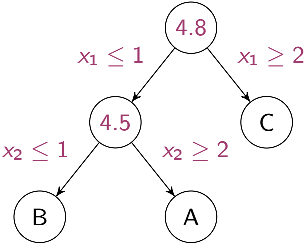
$b_1 = \lfloor \bar{b}_3 \rfloor$

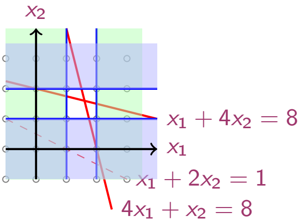
$\bar{b}_1 = \lfloor \bar{b}_3 \rfloor - b_3 < 0$

Dual simplex:

$\min_j |\frac{c_j}{a_i j}|$

- Let's branch again





We have three open problems. Which one we choose next?
Let's take A.

	x1	x2	x3	x4	x5	x6	b	-z
	0	-1	0	0	0	1	0	-2
	0	0	-1/4	1	-15/4		0	9/4
	0	1	15/60	0	-1/4		0	7/4
	1	0	0	0	1		0	1
	0	0	-37/60	0	-9/4		1	-9/2

	x1	x2	x3	x4	x5	x6	b	-z
III+I	0	0	1/4	0	-1/4	1	0	-1/4
	0	0	-1/4	1	-15/4		0	9/4
	0	1	15/60	0	-1/4		0	7/4
	1	0	0	0	1		0	1
	0	0	-37/60	0	-9/4		1	-9/2

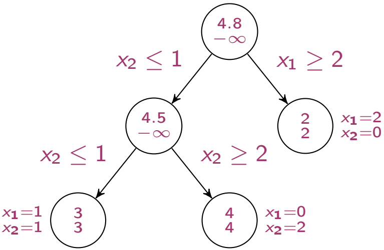
continuing we find:

$x_1 = 0$

$x_2 = 2$

$OPT = 4$

The final tree:

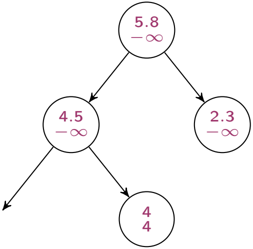


The optimal solution is 4.

Pruning

Pruning:

1. by optimality: $z^k = \max\{c^T x : x \in S^k\}$
2. by bound $\overline{z}^k \leq \underline{z}$
   Example:



3. by infeasibility $S^k = \emptyset$

B&B Components

Bounding:

1. LP relaxation
2. Lagrangian relaxation
3. Combinatorial relaxation
4. Duality

Branching:

$$S_1 = S \cap \{x : x_j \leq \lfloor \bar{x}_j \rfloor\}$$
$$S_2 = S \cap \{x : x_j \geq \lceil \bar{x}_j \rceil\}$$

thus the current optimum is not feasible either in S_1 or in S_2.
Which variable to choose?
Eg: Most fractional variable $\arg\max_{j \in C} \min\{f_j, 1 - f_j\}$

Choosing Node for Examination from the list of active (or open):

- Depth First Search (a good primal sol. is good for pruning + easier to reoptimize by just adding a new constraint)
- Best Bound First: (eg. largest upper: $\bar{z}^s = \max_k \bar{z}^k$ or largest lower - to die fast)
- Mixed strategies

Reoptimizing: dual simplex

Updating the Incumbent: when new best feasible solution is found:

$$\underline{z} = \max\{\underline{z}, 4\}$$

Store the active nodes: bounds + optimal basis (remember the revised simplex!)

Enhancements

- Preprocessor: constraint/problem/structure specific
  tightening bounds
  redundant constraints
  variable fixing: eg: $\max\{c^T x : Ax \leq b, l \leq x \leq u\}$
  fix $x_j = l_j$ if $c_j < 0$ and $a_{ij} > 0$ for all i
  fix $x_j = u_j$ if $c_j > 0$ and $a_{ij} < 0$ for all i
- Priorities: establish the next variable to branch
- Special ordered sets SOS (or generalized upper bound GUB)

$$\sum_{j=1}^{k} x_j = 1 \qquad x_j \in \{0, 1\}$$

instead of: $S_0 = S \cap \{x : x_j = 0\}$ and $S_1 = S \cap \{x : x_j = 1\}$
$\{x : x_j = 0\}$ leaves $k - 1$ possibilities
$\{x : x_j = 1\}$ leaves only 1 possibility
hence tree unbalanced

here: $S_1 = S \cap \{x : x_{j_i} = 0, i = 1..r\}$ and
$S_2 = S \cap \{x : x_{j_i} = 0, i = r + 1, .., k\}$, $r = \min\{t : \sum_{i=1}^{t} x^*_{j_i} \geq \frac{1}{2}\}$

- Cutoff value: a user-defined primal bound to pass to the system.
- Simplex strategies: simplex is good for reoptimizing but for large models interior points methods may work best.
- Strong branching: extra work to decide more accurately on which variable to branch:
  1. choose a set C of fractional variables
  2. reoptimize for each them (in case for limited iterations)
  3. $\overline{z}_j^{\downarrow}, \overline{z}_j^{\uparrow}$ (dual bound of down and up branch)

  $$j^* = \arg\min_{j \in C} \max\{z_j^{\downarrow}, z_j^{\uparrow}\}$$

  ie, choose variable with largest decrease of dual bound, eg UB for max

- If not finished after a certain time, possible reasons:
  - no feasible solution is found
  - the gap best feasible-dual bound is large

$$\text{gap} = \frac{|\text{Primal bound} - \text{Dual bound}|}{\text{Primal bound} + \epsilon} \cdot 100$$

  - runs out of memory

- heuristics for finding feasible solutions (generally NP-complete problem)

- find better lower bounds if they are weak: addition of cuts, stronger formulation, branch and cut

- Branch and cut: a B&B algorithm with cut generation at all nodes of the tree. (instead of reoptimizing, do as much work as possible to tighten)

  Cut pool: stores all cuts centrally
  Store for active node: bounds, basis, pointers to constraints in the cut pool that apply at the node

Relative Optimality Gap

In CPLEX:

$$\text{gap} = \frac{|\text{best node} - \text{best integer}|}{|\text{best integer} + 10^{-11}|}$$

In SCIP and MIPLIB standard:

$$\text{gap} = \frac{pb - db}{\inf\{|z|, z \in [db, pb]\}} \cdot 100 \qquad \text{for a minimization problem}$$

(if $pb \geq 0$ and $db \geq 0$ then $\frac{pb-db}{db}$)
if $db = pb = 0$ then $\text{gap} = 0$
if no feasible sol found or $db \leq 0 \leq pb$ then the gap is not computed.

Last standard avoids problem of non decreasing gap if we go through zero

```
   3186  2520     -666.6217  4096      956.6330     -667.2010  1313338  169.74%
   3226  2560     -666.6205  4097      956.6330     -667.2010  1323797  169.74%
   3266  2600     -666.6201  4095      956.6330     -667.2010  1335602  169.74%
Elapsed real time = 2801.61 sec. (tree size = 77.54 MB, solutions = 2)
*  3324+ 2656                         -125.5775     -667.2010  1363079  431.31%
   3334  2668     -666.5811  4052     -125.5775     -667.2010  1370748  431.31%
   3380  2714     -666.5799  4017     -125.5775     -667.2010  1388391  431.31%
   3422  2756     -666.5791  4011     -125.5775     -667.2010  1403440  431.31%
```

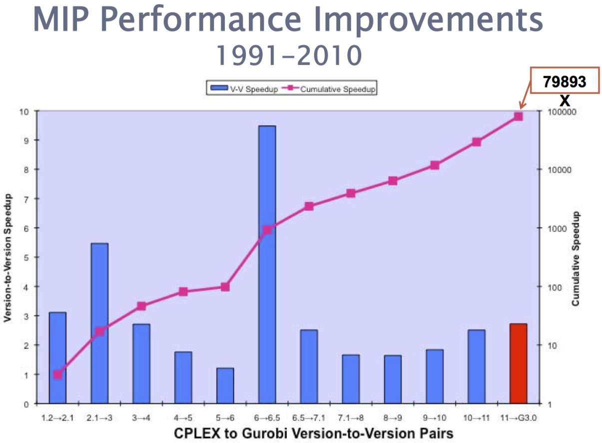
Advanced Techniques

We did not treat:

- LP: Dantzig Wolfe decomposition
- LP: Column generation
- LP: Delayed column generation
- IP: Branch and Price
- LP: Benders decompositions
- LP: Lagrangian relaxation

MILP Solvers Breakthroughs

We have seen Fractional Gomory cuts.
The introduction of Mixed Integer Gomory cuts in CPLEX was the major breakthrough of CPLEX 6.5 and produced the version-to-version speed-up given by the blue bars in the chart below



(source: R. Bixby. Mixed-Integer Programming: It works better than you may think. 2010. Slides on the net)

Summary

1. Cutting Plane Algorithms
2. Branch and Bound