DM545/DM554

Linear and Integer Programming

# Lecture 5
# Duality


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# Outline

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# Dual Problem



Dual variables $\mathbf{y}$ in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad z = \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w = \mathbf{b}^T\mathbf{y} \\ A^T\mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

# Bounding approach



$$\begin{array}{rrrrl} \max & 4x_1 + & x_2 & + 3x_3 & \\ & x_1 & + 4x_2 & & \leq 1 \\ & 3x_1 + & x_2 & + \ x_3 & \leq 3 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

a feasible solution is a lower bound but how good?
By tentatives:

$$(x_1, x_2, x_3) = (1, 0, 0) \rightsquigarrow z^* \geq 4$$
$$(x_1, x_2, x_3) = (0, 0, 3) \rightsquigarrow z^* \geq 9$$

What about upper bounds?

$$\begin{array}{rcrl} & & 2 \cdot (\ \ x_1 + 4x_2 \qquad\quad ) & \leq 2 \cdot 1 \\ & & + 3 \cdot (\ 3x_1 + \ x_2 + \ x_3) & \leq 3 \cdot 3 \\ \hline 4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 11x_2 + 3x_3 & \leq \ 11 \\ \\ c^T x & \leq & y^T A x & \leq y^T b \end{array}$$

Hence $z^* \leq 11$. Is this the best upper bound we can find?

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$
\begin{array}{rcl}
y_1 \cdot (\;\; x_1 + 4x_2 \qquad\qquad\quad ) & \leq & y_1(1) \\
+y_2 \cdot (\; 3x_1 + \;\; x_2 \;+ \qquad\quad x_3) & \leq & y_2(3) \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 & \leq & y_1 + 3y_2
\end{array}
$$

Coefficients

$$
\begin{aligned}
y_1 \;\; + 3y_2 &\geq 4 \\
4y_1 + \;\; y_2 &\geq 1 \\
y_2 &\geq 3
\end{aligned}
$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2$ then to attain the best upper bound:

$$
\begin{aligned}
\min \quad & y_1 + 3y_2 \\
& y_1 + 3y_2 \geq 4 \\
& 4y_1 + y_2 \geq 1 \\
& y_2 \geq 3 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

# Multipliers Approach

$$
\begin{array}{c}
\pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1}
\end{array}
\left[\begin{array}{cccc|cccc|c|c}
a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\
\vdots & \ddots & & & & & & & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 0 & b_m \\
\hdashline
c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0
\end{array}\right]
$$

Working columnwise, since at optimum $\bar{c}_k \leq 0$ for all $k = 1, \dots, n+m$:

$$
\left\{\begin{array}{cccccccc}
\pi_1 a_{11} & + & \pi_2 a_{21} & \dots + & \pi_m a_{m1} & + \pi_{m+1} c_1 & \leq & 0 \\
\vdots & \ddots & & & & & & \vdots \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots + & \pi_m a_{mn} & + \pi_{m+1} c_n & \leq & 0 \\
\hdashline
\pi_1 a_{1,n+1}, & & \pi_2 a_{2,n+1}, & \dots & \pi_m a_{m,n+1} & & \leq & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\pi_1 a_{1,n+m}, & & \pi_2 a_{2,n+m}, & \dots & \pi_m a_{m,n+m} & & \leq & 0 \\
\hdashline
 & & & & & \pi_{m+1} & = & 1 \\
\hdashline
\pi_1 b_1 & + & \pi_2 b_2 & \dots + & \pi_m b_m & & & (\leq 0)
\end{array}\right.
$$

(since from the last row $z = -\pi b$ and we want to maximize $z$ then we would $\min(-\pi b)$ or equivalently $\max \pi b$)

$$\begin{array}{rcrcrcl}
\max \ \pi_1 b_1 & + & \pi_2 b_2 & \ldots + & \pi_m b_m & & \\
\pi_1 a_{11} & + & \pi_2 a_{21} & \ldots + & \pi_m a_{m1} & \leq & -c_1 \\
\vdots & \ddots & & & & & \vdots \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \ldots + & \pi_m a_{mn} & \leq & -c_n \\
& & & & \pi_1, \pi_2, \ldots \pi_m & \leq & 0
\end{array}$$

$y = -\pi$

$$\begin{array}{rcrcrcl}
\max \ -y_1 b_1 & + & -y_2 b_2 & \ldots + & -y_m b_m & & \\
-y_1 a_{11} & + & -y_2 a_{21} & \ldots + & -y_m a_{m1} & \leq & -c_1 \\
\vdots & \ddots & & & & & \vdots \\
-y_1 a_{1n} & + & -y_2 a_{2n} & \ldots + & -y_m a_{mn} & \leq & -c_n \\
& & & & -y_1, -y_2, \ldots - y_m & \leq & 0
\end{array}$$

$$\begin{array}{rrl}
\min & w = & b^T y \\
& A^T y \geq & c \\
& y \geq & 0
\end{array}$$

# Example



$$
\begin{aligned}
\max\ 6x_1 &+ \ 8x_2 \\
5x_1 &+ 10x_2 \leq 60 \\
4x_1 &+ \ 4x_2 \leq 40 \\
x_1, x_2 &\geq \ 0
\end{aligned}
$$

$$
\begin{cases}
5\pi_1 + 4\pi_2 + 6\pi_3 \leq 0 \\
10\pi_1 + 4\pi_2 + 8\pi_3 \leq 0 \\
1\pi_1 + 0\pi_2 + 0\pi_3 \leq 0 \\
0\pi_1 + 1\pi_2 + 0\pi_3 \leq 0 \\
0\pi_1 + 0\pi_2 + 1\pi_3 = 1 \\
60\pi_1 + 40\pi_2
\end{cases}
$$

$$
\begin{aligned}
y_1 &= -\pi_1 \geq 0 \\
y_2 &= -\pi_2 \geq 0
\end{aligned}
$$

...

# Duality Recipe



| | Primal linear program | Dual linear program |
|---|---|---|
| Variables | $x_1, x_2, \ldots, x_n$ | $y_1, y_2, \ldots, y_m$ |
| Matrix | $A$ | $A^T$ |
| Right-hand side | $\mathbf{b}$ | $\mathbf{c}$ |
| Objective function | $\max \mathbf{c}^T\mathbf{x}$ | $\min \mathbf{b}^T\mathbf{y}$ |
| Constraints | $i$th constraint has $\leq$ | $y_i \geq 0$ |
| | $\geq$ | $y_i \leq 0$ |
| | $=$ | $y_i \in \mathbb{R}$ |
| | $x_j \geq 0$ | $j$th constraint has $\geq$ |
| | $x_j \leq 0$ | $\leq$ |
| | $x_j \in \mathbb{R}$ | $=$ |

# Outline

# Symmetry



The dual of the dual is the primal:

Primal problem:

$$\begin{aligned} \max \quad z &= c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= b^T y \\ A^T y &\geq c \\ y &\geq 0 \end{aligned}$$

Let's put the dual in the standard form

Dual problem:

$$\begin{aligned} \min \quad b^T y \quad &\equiv -\max -b^T y \\ -A^T y &\leq -c \\ y &\geq 0 \end{aligned}$$

Dual of Dual:

$$\begin{aligned} -\min c^T x & \\ -Ax &\geq -b \\ x &\geq 0 \end{aligned}$$

# Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

**Theorem (Weak Duality Theorem)**

*Given:*

$$
\begin{aligned}
&(P)\ \max\{c^T x \mid Ax \le b, x \ge 0\}\\
&(D)\ \min\{b^T y \mid A^T y \ge c, y \ge 0\}
\end{aligned}
$$

*for any feasible solution $x$ of (P) and any feasible solution $y$ of (D):*

$$c^T x \le b^T y$$

Proof:

From (D) $c_j \le \sum_{i=1}^m y_i a_{ij}\ \forall j$ and from (P) $\sum_{j=1}^n a_{ij} x_i \le b_i\ \forall i$

From (D) $y_i \ge 0$ and from (P) $x_j \ge 0$

$$\sum_{j=1}^n c_j x_j \le \sum_{j=1}^n \left(\sum_{i=1}^m y_i a_{ij}\right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_i\right) y_i \le \sum_{i=1}^m b_i y_i$$

# Strong Duality Theorem

Due to Von Neumann and Dantzig 1947 and Gale, Kuhn and Tucker 1951.

**Theorem (Strong Duality Theorem)**

*Given:*

$$(P)\ \max\{c^T x \mid Ax \leq b, x \geq 0\}$$
$$(D)\ \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

*exactly one of the following occurs:*

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution* $x^* = [x_1^*, \ldots, x_n^*]$
   *(D) has feasible solution* $y^* = [y_1^*, \ldots, y_m^*]$

$$c^T x^* = b^T y^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.

- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)

- The last row of the final tableau will give us

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k = z^* + \sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i=1}^{m} \bar{c}_{n+i} x_{n+i} \qquad (*)$$

$$= z^* + \bar{c}_B x_B + \bar{c}_N x_N$$

  In addition, $z^* = \sum_{j=1}^{n} c_j x_j^*$ because optimal value

- We define $\boxed{y_i^* = -\bar{c}_{n+i},}\ i = 1, 2, \ldots, m$

- We claim that $(y_1^*, y_2^*, \ldots, y_m^*)$ is a dual feasible solution satisfying $c^T x^* = b^T y^*$.

- Let's verify the claim:
We substitute in (*): $z = \sum c_j x_j$, $\bar{c}_{n+1} = -y_i^*$ and $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, 2, \ldots, m$ ($n + i$ are the slack variables)

$$\sum c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$
$$= \left( z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left( \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

This must hold for every $(x_1, x_2, \ldots, x_n)$ hence:

$$z^* = \sum_{i=1}^{m} b_i y_i^* \qquad \Longrightarrow\ y^* \text{ satisfies } c^T x^* = b^T y^*$$

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, j = 1, 2, \ldots, n$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \ldots, n + m$:

$$\bar{c}_j \leq 0 \rightsquigarrow \quad c_j - \sum_{i=1}^{m} y_i^* a_{ij} \leq 0 \rightsquigarrow \quad \sum_{i=1}^{m} y_i^* a_{ij} \geq c_j \qquad j = 1, 2, \ldots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow \quad y_i^* = -\hat{c}_{n+i} \geq 0, \qquad i = 1, 2, \ldots, m$$

$\implies$ $y^*$ is also dual feasible solution

# Complementary Slackness Theorem



**Theorem (Complementary Slackness)**

*A feasible solution $x^*$ for (P)*
*A feasible solution $y^*$ for (D)*
*Necessary and sufficient conditions for optimality of both:*

$$\left(c_j - \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j^* = 0, \quad j = 1, \ldots, n$$

*If $x_j^* \neq 0$ then $\sum y_i^* a_{ij} = c_j$ (no surplus)*
*If $\sum y_i^* a_{ij} > c_j$ then $x_j^* = 0$*

Proof:

$$z^* = c^T x^* \leq y^* A x^* \leq b^T y^* = w^*$$

Hence from strong duality theorem:

$$cx^* - yAx^* = 0$$

In scalars

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be $= 0$

# Duality - Summary

- Derivation:
  - Economic interpretation
  - Bounding Approach
  - Multiplier Approach
  - Recipe
  - Lagrangian Multipliers Approach (next time)
- Theory:
  - Symmetry
  - Weak Duality Theorem
  - Strong Duality Theorem
  - Complementary Slackness Theorem