DM545

Linear and Integer Programming

Lecture 8

More on Polyhedra and Farkas Lemma

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

LP: Rational Solutions

- A precise analysis of running time for an algorithm includes the number of bit operations together with the number of arithmetic operations.

Example

The knapsack problem aka, budget allocation problem, that asks to choose amont a set of n investments those that maximize the profit and cost in total less than B, can be solved by dynamic programming in

$$O(n|B|)$$

The number B needs $b = \log |B|$ bits hence the running time is exponential in the number of bits needed to represent B, ie, $O(n2^b)$

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- Weakly polynomial time algorithms have running time that are independent on the sizes of the numbers involved in the problem and hence on the number of bits needed to represent them.
- Strongly polynomial time algorithms: the running time of the algorithm is independent on the number of bit operations. Eg: same running time for input numbers with 10 bits as for inputs with a million bits.

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- In spite of this: No strongly polynomial-time algorithm for LP is known.

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3. Transform the feasible region to place the current point at the center of it

- because of patents reasons, now mostly known as barrier algorithms
- one single iteration is computationally more intensive than the simplex (matrix calculations, sizes depend on number of variables)
- particularly competitive in presence of many constraints (eg, for $m = 10,000$ may need less than 100 iterations)
- bad for post-optimality analysis $\rightsquigarrow$ crossover algorithm to convert a sol of barrier method into a basic feasible solution for the simplex

How Large Problems Can We Solve?

Very large model

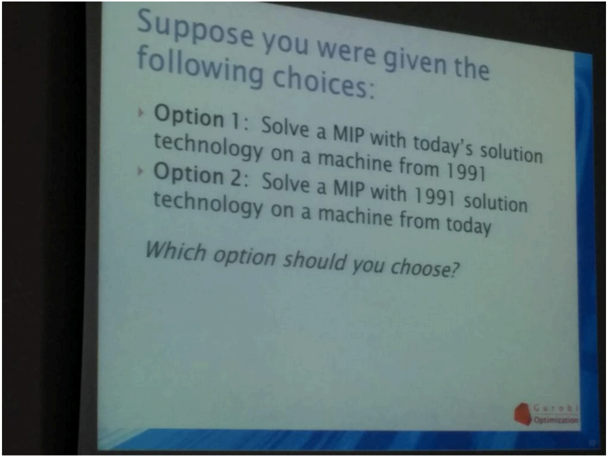
	Rows	Columns	Nonzeros
Original size	5034171	7365337	25596099
After presolve	1296075	2910559	10339042

Solution times were as follows:

Very large model—solution times

	Algorithm		
Version	Barrier	Dual	Primal
CPLEX 5.0	8642.6	350000.0	71039.7
CPLEX 7.1	5642.6	6413.1	1880.0

Source: Bixby, 2002



Marco Lübbecke @mluebbecke · Apr 18
hint: option 1 is correct #orms #math #algorithms

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2. Farkas Lemma

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3. Intersection of n hyperplanes in n dimensions: when do they uniquely identify a point?
   when the rank of the matrix A of the linear system is n (or A is nonsingular)

Vertices of Polyhedra

A vertex of a polyhedron is a point that is a feasible solution to the system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}
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   Yes, just look at the pyramid in 3 dim. Rank of the matrix of active constraints is still n

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   the number of possible active constraints is $\binom{m}{n}$
   it is an upper bound because:
   - some combinations of constraints will not define a vertex, ie, if rows of matrix not independent
   - some vertices may activate more than n constraints and hence the same vertex can be given by more than n constraints

Tableaux and Vertices

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	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
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3 constraints. With slack variables we would have 6 variables in all, if any of them is positive the constraint $x_i \geq 0$ of the original variables would be active, otherwise the corresponding constraint of the original problem are active.

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For what seen above this translates in $n-1$ variables in common in the tableau

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2. Farkas Lemma

We now look at Farkas Lemma with two objectives:

- giving another proof of strong duality
- understanding a certificate of infeasibility

Farkas Lemma

Lemma (Farkas)

Let $A \in \mathbb{R}^{m \times n}$ *and* $\mathbf{b} \in \mathbb{R}^m$. *Then,*

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or	*II.*	$\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y}^T A \geq 0^T$ *and* $\mathbf{y}^T \mathbf{b} < \mathbf{0}$

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Easy to see that both I and II cannot occur together:

$$A\mathbf{x} = \quad \mathbf{b}$$

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$$(0 \leq) \qquad \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b} \qquad (< 0)$$

Geometric interpretation of Farkas L.

Linear combination of $\mathbf{a}_i$ with nonnegative terms generates a convex cone:

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Polyhedral cone: $C = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, intersection of many $\mathbf{ax} \leq 0$
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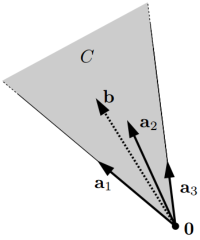

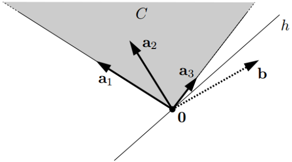

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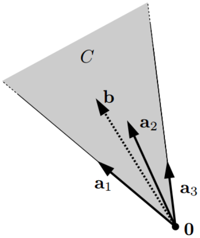

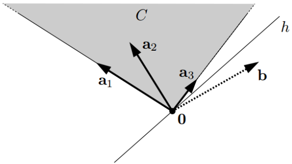

Either point $\mathbf{b}$ lies in convex cone C

or $\exists$ hyperplane h passing through point 0 $h = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{y}^T\mathbf{x} = 0\}$ for $\mathbf{y} \in \mathbb{R}^m$ such that all vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ (and thus C) lie on one side and $\mathbf{b}$ lies (strictly) on the other side (ie, $\mathbf{y}^T\mathbf{a}_i \geq 0, \forall i = 1 \ldots n$ and $\mathbf{y}^T\mathbf{b} < 0$).

Variants of Farkas Lemma

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(i) $A\mathbf{x} = \mathbf{b}$ *has sol* $\mathbf{x} \geq \mathbf{0} \iff \forall \mathbf{y} \in \mathbb{R}^m$ *with* $\mathbf{y}^T A \geq \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq 0$

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relation with Fourier & Moutzkin method

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Strong Duality by Farkas Lemma

$$(P) \qquad \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

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Let's define:

$$\hat{A} = \begin{bmatrix} A \\ -\mathbf{c}^T \end{bmatrix} \qquad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ -\gamma - \epsilon \end{bmatrix}$$

and consider $\hat{A}\mathbf{x} \leq \hat{\mathbf{b}}_0$ and $\hat{A}\mathbf{x} \leq \hat{\mathbf{b}}_\epsilon$

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By weak duality γ is lower bound for D. Since D bounded and feasible then there exists $\mathbf{y}^*$:

$$\gamma \leq \mathbf{b}^T\mathbf{y}^* < \gamma + \epsilon \qquad \forall \epsilon > 0$$

which implies $\mathbf{b}^T\mathbf{y}^* = \gamma$

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Farkas Lemma provides a way to certificate infeasibility.

Theorem

Given a certificate $\mathbf{y}^*$ *it is easy to check the conditions (by linear algebra):*

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$$(\mathbf{y}^*)^T A\mathbf{x}^* = (\mathbf{y}^*)^T\mathbf{b}$$

Certificate of Infeasibility

Farkas Lemma provides a way to certificate infeasibility.

Theorem

Given a certificate $\mathbf{y}^*$ *it is easy to check the conditions (by linear algebra):*

$$A^T\mathbf{y}^* \geq \mathbf{0}$$
$$\mathbf{b}\mathbf{y}^* < 0$$

Why would $\mathbf{y}^*$ be a certificate of infeasibility?
Proof (by contradiction)
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Contradiction

General form:

$$\begin{aligned} \max\ & c^T x \\ A_1 x &= b_1 \\ A_2 x &\leq b_2 \\ A_3 x &\geq b_3 \\ x &\geq 0 \end{aligned}$$

infeasible $\Leftrightarrow \exists y^*$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 + b_3^T y_3 &> 0 \\ A_1^T y_1 + A_2^T y_2 + A_3^T y_3 &\leq 0 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \end{aligned}$$

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Example

$$\begin{aligned} \max\ & c^T x \\ x_1 &\leq 1 \\ x_1 &\geq 2 \end{aligned}$$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 &> 0 \\ A_1^T y_1 + A_2^T y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 &> 0 \\ y_1 + y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$y_1 = -1, y_2 = 1$ is a valid certificate.

- Observe that it is not unique!
- It can be reported in place of the dual solution because same dimension.
- To repair infeasibility we should change the primal at least so much as that the certificate of infeasibility is no longer valid.
- Only constraints with $y_i \neq 0$ in the certificate of infeasibility cause infeasibility

Duality: Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
  - Farkas Lemma:
    Strong duality + Infeasibility certificate
- Dual Simplex
- Economic interpretation
- Geometric Interpretation
- Sensitivity analysis

Resume

Advantages of considering the dual formulation:

- proving optimality (although the simplex tableau can already do that)
- gives a way to check the correctness of results easily
- alternative solution method (ie, primal simplex on dual)
- sensitivity analysis
- solving P or D we solve the other for free
- certificate of infeasibility