DM554

Linear and Integer Programming

Lecture 4
Systems of Linear Equations

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Outline

Outline

1. Solving Linear Systems

2. Elementary Matrices

Problem Statement

Given the system of linear equations:

$$\begin{array}{l|rcrcrcl} \texttt{I:} & x_1 & + & x_2 & + & x_3 & = & 3 \\ \texttt{II:} & 2x_1 & + & x_2 & + & x_3 & = & 4 \\ \texttt{III:} & x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array}$$

Find whether it has any solution and in case characterize the solutions.

Augmented Matrix

Definition (Augmented Matrix and Elementary row operations)

For a system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

the augmented matrix of the system and the row operations are:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

RO1: multiply a row by a non-zero constant

RO2: interchange two rows

RO3: add a multiple of one row to another

They modify the linear system into an equivalent system (same solutions)

Gaussian Elimination: Example

Let's consider the system $A\mathbf{x} = \mathbf{b}$ with:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}$$

1. Left most column that is not all zeros (it is column 1)
2. A non-zero entry at the top of this column (it is the one on the top)
3. Make the entry 1 (it is already)

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}$$

4. make all entries below the leading one zero:

$$\begin{matrix} \texttt{I'=I} \\ \texttt{II'=II-2I} \\ \texttt{III'=III-I} \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & 1 & 2 \end{bmatrix}$$

Example, cntd. Row Echelon Form

5. Cover up the top row and apply steps (1) and (4) again
1. Left most column that is not all zeros is column 2
2. Non-zero entry at the top of the column
3. Make this entry the leading 1 by elementary row operations RO1 or RO2.
4. Make all entries below the leading 1 zero by RO3

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} & \cancel{3} \\ 0 & -1 & -1 & -2 \\ 0 & -2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 3 & 6 \end{bmatrix} \equiv \begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 3 \\ & & x_2 & + & x_3 & = & 2 \\ & & & & x_3 & = & 2 \end{array}$$

Definition (Row echelon form)

A matrix is said to be in row echelon form (or echelon form) if it has the following three properties:

1. the first nonzero entry in each nonzero row is 1
2. a leading 1 in a lower row is further to the right
3. zero rows are at the bottom of the matrix

Back substitution

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 3 \\ & & x_2 & + & x_3 & = & 2 \\ & & & & x_3 & = & 2 \end{array}$$

From the row echelon form we solve the system by back substitution:

- from the last equation: set $x_3 = 2$
- substitute x_3 in the second equation $\rightsquigarrow x_2$
- substitute x_2 and x_3 in the first equation $\rightsquigarrow x_1$

Reduced Row Echelon Form

In the augmented matrix representation:

6. Begin with the last row and add suitable multiples to each row above to get zero above the leading 1.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Definition (Reduced row echelon form)

A matrix is said to be in reduced (row) echelon form if it has the following properties:

1. The matrix is in row echelon form
2. Every column with a leading 1 has zeros elsewhere

From a Reduced Row Echelon Form (RREF) we can read the solution:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The system has a unique solution.
Is it a correct solution? Let's check:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Gaussian Elimination: Algorithm

Gaussian Elimination algorithm for solving a linear system:
(puts the augmented matrix in a form from which the solution can be read)

1. Find left most column that is not all zeros
2. Get a non-zero entry at the top of this column (pivot element)
3. Make this entry 1 by elementary row operations RO1 or RO2. This entry is called leading one
4. Add suitable multiples of the top row to rows below so that all entries below the leading one become zero
5. Cover up the top row and apply steps (1) and (4) again
   The matrix left is in (row) echelon form
6. Back substitution

Gauss-Jordan Reduction

Gauss Jordan Reduction algorithm for solving a linear system:
(puts the augmented matrix in a form from which the solution can be read)

1. Find left most column that is not all zeros
2. Get a non-zero entry at the top of this column (pivot element)
3. Make this entry 1 by elementary row operations RO1 or RO2. This entry is called leading one
4. Add suitable multiples of the top row to rows below so that all entries below the leading one become zero
5. Cover up the top row and apply steps (1) and (4) again
   The matrix left is in (row) echelon form
6. Begin with the last row and add suitable multiples to each row above to get zero above the leading 1.
   The matrix left is in reduced (row) echelon form

Will there always be exactly one solution?

$$\begin{array}{l|rcrcl} \texttt{i:} & & & 2x_3 & = & 3 \\ \texttt{ii:} & 2x_2 & + & 3x_3 & = & 4 \\ \texttt{iii:} & & & x_3 & = & 5 \end{array} \quad \to \quad \left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{array}{r} \texttt{ii} \\ \texttt{i} \\ \\ \end{array} \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{array}{r} \texttt{i/2} \\ \\ \\ \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to$$

$$\to \begin{array}{r} \\ \texttt{iii} \\ \texttt{ii} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 2 & 3 \end{bmatrix} \to \begin{array}{r} \\ \\ \texttt{iii-2ii} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix} \to$$

$$\to \begin{array}{r} \\ \\ \texttt{-iii/7} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

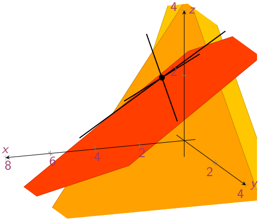
No Solution!

Definition (Consistent)

A system of linear equations is said to be consistent if it has at least one solution. It is inconsistent if there are no solutions.

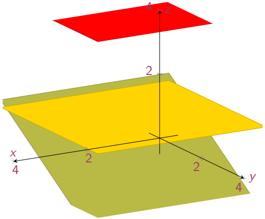
$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}$$

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}$$



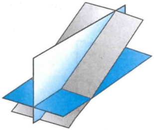
$$\begin{cases} 2x_3 = 3 \\ 2x_2 + 3x_3 = 4 \\ x_3 = 5 \end{cases}$$

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

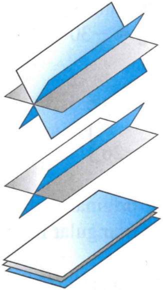


Geometric Interpretation

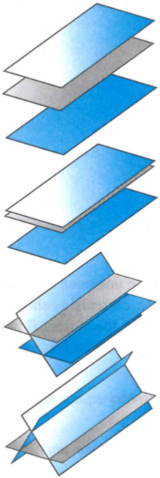
Three equations in three unknowns interpreted as planes in space



Unique solution



Infinitely
many solutions



No solution

Definition (Overdetermined)

A linear system is said to be over-determined if there are more equations than unknowns. Over-determined systems are usually (but not always) inconsistent.

Definition (Underdetermined)

A linear system of m equations and n unknowns is said to be under-determined if there are fewer equations than unknowns ($m < n$). They have usually infinitely many solutions (never just one).

Linear systems with free variables

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 3 \\ 2x_1 + x_2 + x_3 + x_4 + 2x_5 &= 4 \\ x_1 - x_2 - x_3 + x_4 + x_5 &= 5 \\ x_1 \qquad\qquad + x_4 + x_5 &= 4 \end{aligned}$$

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & 1 & 1 & 5 \\ 1 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} \\ \texttt{ii-2i} \\ \texttt{iii-i} \\ \texttt{iv-i} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -1 & 0 & -2 \\ 0 & -2 & -2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} \\ \texttt{(-1)ii} \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & -2 & -2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} \\ \\ \texttt{iii+2ii} \\ \texttt{iv+ii} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} \\ \\ \texttt{(1/2)iii} \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} \\ \\ \\ \texttt{iv-iii} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row echelon form}$$

$$\rightarrow \quad \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \texttt{i-iii} \\ \texttt{ii-iii} \\ \rightarrow \\ \end{array} \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \texttt{i-ii} \\ \\ \rightarrow \\ \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccccccccccc} x_1 & + & 0 & + & 0 & + & 0 & + & x_5 & = & 1 \\ & + & x_2 & + & x_3 & + & 0 & + & 0 & = & -1 \\ & & & & & + & x_4 & + & 0 & = & 3 \end{array}$$

$$\begin{array}{ccccccccccc} x_1 & + & 0 & + & 0 & + & 0 & + & x_5 & = & 1 \\ & + & x_2 & + & x_3 & + & 0 & + & 0 & = & -1 \\ & & & & & + & x_4 & + & 0 & = & 3 \end{array}$$

Definition (Leading variables)

The variables corresponding with leading ones in the reduced row echelon form of an augmented matrix are called leading variables. The other variables are called non-leading variables

- x_1, x_2 and x_4 are leading variables.
- x_3, x_5 are non-leading variables.
- we assign x_3, x_5 the arbitrary values $s, t \in \mathbb{R}$ and solve for the leading variables.
- there are infinitely many solutions, represented by the general solution:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1-t \\ -1-s \\ s \\ 3 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution Sets

Theorem

A system of linear equations either has no solutions, a unique solution or infinitely many solutions.

Proof.

Let's assume the system $A\mathbf{x} = \mathbf{b}$ has two solutions $\mathbf{p}$ and $\mathbf{q}$. Then all points on the line connecting these two points are also solutions and so there are infinitely many solutions.

$$A\mathbf{p} = \mathbf{b} \qquad \mathrm{A}\mathbf{q}=\mathbf{b} \qquad \mathbf{p} \neq \mathbf{q}$$

$$\mathbf{v} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}), t \in \mathbb{R}$$

$$A\mathbf{v} = A(\mathbf{p} + t(\mathbf{q} - \mathbf{p})) = A\mathbf{p} + t(A\mathbf{q} - A\mathbf{p}) = \mathbf{b} + t(\mathbf{b} - \mathbf{b}) = \mathbf{b}$$

Homogeneous systems

Definition (Homogenous system)

An homogeneous system of linear equations is a linear system of the form $A\mathbf{x} = \mathbf{0}$.

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ is always consistent $A\mathbf{0} = \mathbf{0}$.
- If $A\mathbf{x} = \mathbf{0}$ has a unique solution, then it must be the trivial solution $\mathbf{x} = \mathbf{0}$.

In the augmented matrix the last column stays always zero $\rightsquigarrow$ we can omit it.

Example

$$\begin{array}{rcrcrcrcl} x & + & y & + & 3z & + & w & = & 0 \\ x & - & y & + & z & + & w & = & 0 \\ & & y & + & 2z & + & 2w & = & 0 \end{array} \qquad A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Theorem

If A is an $m \times n$ matrix with $m < n$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Proof.

- The system is always consistent since homogeneous.
- Matrix A brought in reduced echelon form contains at most m leading ones (variables).
- $n - m \geq 1$ non-leading variables

How about $A\mathbf{x} = \mathbf{b}$ with A $m \times n$ and $m < n$?
If the system is consistent, then there are infinitely many solutions.

Example

$$\begin{array}{ccccccccc} x & + & y & + & 3z & + & w & = & 2 \\ x & - & y & + & z & + & w & = & 4 \\ & & y & + & 2z & + & 2w & = & 0 \end{array}$$

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \quad t \in$$

$$A\mathbf{x} = \mathbf{0}$$
$$RREF(A)$$
$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$
$$RREF([A|\mathbf{b}])$$
$$\begin{bmatrix} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Definition (Associated homogenous system)

Given a system of linear equations, $A\mathbf{x} = \mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{0}$ is called the associated homogeneous system

Eg:

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

How can you tell from here that $A\mathbf{x} = \mathbf{b}$ is consistent with infinitely many solutions?

Definition (Null space)

For an $m \times n$ matrix A, the null space of A is the subset of $\mathbb{R}^n$ given by

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

where $\mathbf{0} = (0, 0, \ldots, 0)^T$ is the zero vector of $\mathbb{R}^n$

Theorem (Principle of Linearity)

Suppose that A is an $m \times n$ matrix, that $\mathbf{b} \in \mathbb{R}^m$ and that the system $A\mathbf{x} = \mathbf{b}$ is consistent. Suppose that $\mathbf{p}$ is any solution of $A\mathbf{x} = \mathbf{b}$. Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ consists precisely of the vectors $\mathbf{p} + \mathbf{z}$ for $\mathbf{z} \in N(A)$; ie,

$$\{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\} = \{\mathbf{p} + \mathbf{z} \mid \mathbf{z} \in N(A)\}.$$

Proof: We show that

1. $\mathbf{p} + \mathbf{z}$ is a solution for any $\mathbf{z}$ in the null space of A $(\{\mathbf{p} + \mathbf{z} \mid \mathbf{z} \in N(A)\} \subseteq \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\})$
2. that all solutions, $\mathbf{x}$, of $A\mathbf{x} = \mathbf{b}$ are of the form $\mathbf{p} + \mathbf{z}$ for some $\mathbf{z} \in N(A)$ $(\{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\} \subseteq \{\mathbf{p} + \mathbf{z} \mid \mathbf{z} \in N(A)\})$

1. $A(\mathbf{p} + \mathbf{z}) = A\mathbf{p} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$ so $\mathbf{p} + \mathbf{z} \in \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\}$

2. Let x be a solution. Because $\mathbf{p}$ is also we have $A\mathbf{p} = \mathbf{b}$ and$A(\mathbf{x} - \mathbf{p}) = A\mathbf{x} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$ so $\mathbf{z} = \mathbf{x} - \mathbf{p}$ is a solution of $A\mathbf{z} = \mathbf{0}$ and $\mathbf{x} = \mathbf{p} + \mathbf{z}$

(Check validity of the theorem on the previous examples.)

Summary

- If $A\mathbf{x} = \mathbf{b}$ is consistent, the solutions are of the form:

$$\{\text{solutions of } A\mathbf{x} = \mathbf{b}\} = \mathbf{p} + \{\text{solutions of } A\mathbf{x} = \mathbf{0}\}$$

  - if $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - if $A\mathbf{x} = \mathbf{b}$ has a infinitely many solutions, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions

- $A\mathbf{x} = \mathbf{b}$ may be inconsistent, but $A\mathbf{x} = \mathbf{0}$ is always consistent.

Outline

Matrix Inverse

Let's examine the process of applying the elementary row operations:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \overrightarrow{\mathbf{a}}_2 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix}$$

($\overrightarrow{\mathbf{a}}_i$ row ith of matrix A)
Then the three operations can be described as:

$$\begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \lambda\overrightarrow{\mathbf{a}}_2 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix} \qquad \begin{bmatrix} \overrightarrow{\mathbf{a}}_2 \\ \overrightarrow{\mathbf{a}}_1 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix} \qquad \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \overrightarrow{\mathbf{a}}_2 + \lambda\overrightarrow{\mathbf{a}}_1 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix}$$

For any $n \times n$ matrices A and B:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 B \\ \overrightarrow{\mathbf{a}}_2 B \\ \vdots \\ \overrightarrow{\mathbf{a}}_n B \end{bmatrix}$$

$$\begin{bmatrix} \overrightarrow{\mathbf{a}}_1 B \\ \overrightarrow{\mathbf{a}}_2 B + \lambda \overrightarrow{\mathbf{a}}_1 B \\ \vdots \\ \overrightarrow{\mathbf{a}}_n B \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 B \\ (\overrightarrow{\mathbf{a}}_2 + \lambda \overrightarrow{\mathbf{a}}_1) B \\ \vdots \\ \overrightarrow{\mathbf{a}}_n B \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \overrightarrow{\mathbf{a}}_2 + \lambda \overrightarrow{\mathbf{a}}_1 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix} B$$

(matrix obtained by a row operation on AB)

= (matrix obtained by a row operation on A)B

(matrix obtained by a row operation on B)

= (matrix obtained by a row operation on I)B

Elementary matrix

Definition (Elementary matrix)

An elementary matrix, E, is an $n \times n$ matrix obtained by doing exactly one row operation on the $n \times n$ identity matrix, I.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \quad \xrightarrow{ii-i} \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \xrightarrow{ii-i} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$E_1 B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$