DM554

Linear and Integer Programming

Lecture 5

Matrix Inverse and Determinants

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Elementary Matrices
2. Matrix Inverse
3. Determinants
4. Matrix Inverse and Cramer's rule

Outline

1. Elementary Matrices

2. Matrix Inverse

3. Determinants

4. Matrix Inverse and Cramer's rule

Elementary matrix

Definition (Elementary matrix)

An elementary matrix, E, is an $n \times n$ matrix obtained by doing exactly one row operation on the $n \times n$ identity matrix, I.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{ii-i} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{ii-i} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$E_1 B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Matrix Inverse

The three elementary row operations are trivially invertible.

Theorem

Any elementary matrix is invertible, and the inverse is also an elementary matrix

$$E_1 B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1}(E_1 B) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$$

Row equivalence

To be an equivalence relation a relation must satisfy three properties:

- reflexive: $A \sim B$
- symmetric: $A \sim B \implies B \sim A$
- transitive: $A \sim B$ and $B \sim C \implies A \sim C$

Definition (Row equivalence)

If two matrices A and B are $m \times n$ matrices, we say that A is row equivalent to B if and only if there is a sequence of elementary row operations to transform A to B.

Theorem

Every matrix is row equivalent to a matrix in reduced row echelon form

Invertible Matrices

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A^{-1} *exists*
2. $A\mathbf{x} = \mathbf{b}$ *has a unique solution for any* $\mathbf{b} \in \mathbb{R}^n$
3. $A\mathbf{x} = \mathbf{0}$ *only has the trivial solution,* $\mathbf{x} = \mathbf{0}$
4. *The reduced row echelon form of* A *is* I.

Proof: $(1) \implies (2) \implies (3) \implies (4) \implies (1)$.

- $(1) \implies (2)$

  $$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \implies I\mathbf{x} = A^{-1}\mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$$

  hence $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution and it is unique, indeed:
  $A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}, \quad \forall \mathbf{b}$

- $(2) \implies (3)$
  If $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$, then this is true for $\mathbf{b} = 0$. The unique solution of $A\mathbf{x} = \mathbf{0}$ must be the trivial solution, $\mathbf{x} = \mathbf{0}$

- (3) $\implies$ (4)
  then in the reduced row echelon form of A there are no non-leading (free) variables and there is a leading one in every column hence also a leading one in every row (because A is square and in RREF) hence it can only be the identity matrix

- (4) $\implies$ (1)
  $\exists$ sequence of row operations and elementary matrices $E_1, \ldots, E_r$ that reduce A to I ie,

  $$E_r E_{r-1} \cdots E_1 A = I$$

  Each elementary matrix has an inverse hence multiplying repeatedly on the left by E_r^{-1}, E_{r-1}^{-1}:

  $$A = E_1^{-1} \cdots E_{r-1}^{-1} E_r^{-1} I$$

  hence, A is a product of invertible matrices hence invertible.
  (Recall that $B^{-1}A^{-1} = (AB)^{-1}$)

Outline

Matrix Inverse via Row Operations

We saw that:

$$A = E_1^{-1} \cdots E_{r-1}^{-1} E_r^{-1} I$$

taking the inverse of both sides:

$$A^{-1} = (E_1^{-1} \cdots E_{r-1}^{-1} E_r^{-1})^{-1} = E_r \cdots E_1 = E_r \cdots E_1 I$$

Hence:

$$\text{if} \quad E_r E_{r-1} E \cdots E_1 A = I \quad \text{then} \quad A^{-1} = E_r E_{r-1} \cdots E_1 I$$

Method:

- Construct $[A \mid I]$
- Use row operations to reduce this to $[I \mid B]$
- If this is not possible then the matrix is not invertible
- If it is possible then $B = A^{-1}$

Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \to [A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[iii+i]{ii-i} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{iii-2ii} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \xrightarrow[ii-2iii]{i-4iii} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -11 & 8 & -4 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{i-2ii} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -7 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Verify by checking $AA^{-1} = I$ and $A^{-1}A = I$.
What would happen if the matrix is not invertible?

Verifying an Inverse

Theorem

If A and B are $n \times n$ matrices and $AB = I$, then A and B are each invertible matrices, and $A = B^{-1}$ and $B = A^{-1}$.

Proof: show that $B\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = 0$, then B is invertible.

$$B\mathbf{x} = \mathbf{0} \implies A(B\mathbf{x}) = A\mathbf{0} \implies (AB)\mathbf{x} = \mathbf{0} \stackrel{AB=I}{\implies} I\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$$

So B^{-1} exists for the previous theorem. Hence:

$$AB = I \implies (AB)B^{-1} = IB^{-1} \implies A(BB^{-1}) = B^{-1} \implies A = B^{-1}$$

So A is the inverse of B, and therefore also invertible and

$$A^{-1} = (B^{-1})^{-1} = B$$

Outline

Determinants

- The determinant of a matrix A is a particular number associated with A, written $|A|$ or $\det(A)$, that tells whether the matrix A is invertible.

- For the 2×2 case:

$$[A \mid I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{(1/a)R_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - cb/a & -c/a & 1 \end{array}\right] \xrightarrow{aR_2} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & (ad - bc) & -c & a \end{array}\right]$$

Hence A^{-1} exists if and only if $ad - bc \neq 0$.

- hence, for a 2×2 matrix the determinant is

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- The extension to $n \times n$ matrices is done recursively

Definition (Minor)

For an $n \times n$ matrix the (i,j) minor of A, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and the jth column of A.

Definition (Cofactor)

The (i,j) cofactor of a matrix A is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Definition (Cofactor Expansion of $|A|$ by row one)

The determinant of an $n \times n$ matrix is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 1 \\ -1 & 3 & 0 \end{bmatrix}$$

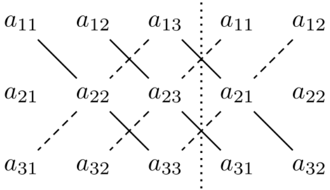
$$\begin{aligned} |A| &= 1C_{11} + 2C_{12} + 3C_{13} \\ &= 1\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ -1 & 3 \end{vmatrix} \\ &= 1(-3) - 2(1) + 3(13) = 34 \end{aligned}$$

Theorem

If A is an $n \times n$ matrix, then the determinant of A can be computed by multiplying the entries of any row (or column) by their cofactors and summing the resulting products:

$$\begin{aligned} |A| =& a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ & \textit{(cofactor expansion by row } i\textit{)} \\ |A| =& a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \\ & \textit{(cofactor expansion by column } j\textit{)} \end{aligned}$$

A mnemonic rule for the 3×3 matrix determinant: the rule of Sarrus



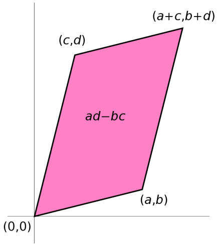
$$
\begin{aligned}
|A| = & + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
& - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{aligned}
$$

Verify the rule:

- from the conditions of existence of an inverse
- as a consequence of the general recursive rule for the determinants

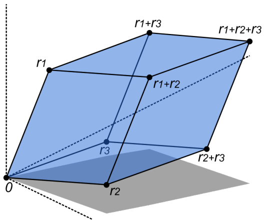
Geometric interpretation

2×2



The area of the parallelogram is the absolute value of the determinant of the matrix formed by the vectors representing the parallelogram's sides.

3×3



The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the rows constructed from the vectors $r1$, $r2$, and $r3$.

Properties of Determinants

Let A be an $n \times n$ matrix, then it follows from the previous theorem:

1. $|A^T| = |A|$
2. If a row of A consists entirely of zeros, then $|A| = 0$.
3. If A contains two rows which are equal, then $|A| = 0$.

$$|A| = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = -d \begin{vmatrix} b & c \\ b & c \end{vmatrix} + e \begin{vmatrix} a & c \\ a & c \end{vmatrix} - f \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 + 0 + 0$$

For 1. we can substitute row with column in 2., 3., 4.

4. If the cofactors of one row are multiplied by the entries of a different row and added, then the result is 0. That is, if $i \neq j$, then $a_{j1}C_{i1} + a_{j2}C_{i2} + \cdots + a_{jn}C_{in} = 0$.

$$A = \begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{matrix} \\ i\text{th} \\ \\ \\ \\ \end{matrix} \qquad |A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$B = \begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{matrix} \\ i\text{th} \\ \\ \\ \\ \end{matrix} \qquad |B| = a_{j1}C_{i1} + a_{j2}C_{i2} + \cdots + a_{jn}C_{in} = 0$$

5. If $A = (a_{ij})$ and if each entry of one of the rows, say row i, can be expressed as a sum of two numbers, $a_{ij} = b_{ij} + c_{ij}$ for $i \leq j \leq n$, then $|A| = |B| + |C|$, where B is the matrix A with row i replaced by $b_{i1}, b_{i2}, \cdots, b_{in}$ and C is the matrix A with row i replaced by $c_{i1}, c_{i2}, \cdots, c_{in}$.

$$|A| = \begin{vmatrix} a & b & c \\ d+p & e+q & f+r \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ g & h & i \end{vmatrix} = |B| + |C|$$

Triangular Matrices

Definition (Triangular Matrices)

An $n \times n$ matrix is said to be upper triangular if $a_{ij} = 0$ for $i > j$ and lower triangular if $a_{ij} = 0$ for $i < j$. Also A is said to be triangular if it is either upper triangular or lower triangular.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition (Diagonal Matrices)

An $n \times n$ matrix is diagonal if $a_{ij} = 0$ whenever $i \neq j$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Determinant using row operations

- Which row or column would you choose for the cofactor expansion in this case:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} ? = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

- if A is upper/lower triangular or diagonal, then $|A| = a_{11}a_{22}\cdots a_{nn}$
- Idea: a square matrix in REF is upper triangular. What is the effect of row operations on the determinant?

RO1 multiply a row by a non-zero constant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad |B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$|B| = \alpha a_{i1}C_{i1} + \alpha a_{i2}C_{i2} + \cdots + \alpha a_{in}C_{in} = \alpha|A|$$

$\rightsquigarrow$ $|A|$ changes to $\alpha|A|$

RO2 interchange two rows

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = 0 \qquad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = 0 \implies |B| = -|A|$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad |B| = \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \implies |B| = -|A|$$

$\rightsquigarrow$ $|A|$ changes to $-|A|$ (by induction)

RO3 add a multiple of one row to another

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad |B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} + 4a_{11} & a_{22} + 4a_{12} & \cdots & a_{2n} + 4a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\begin{aligned} |B| =& (a_{j1} + \lambda a_{i1})C_{j1} + (a_{j2} + \lambda a_{i2})C_{j2} + \cdots + (a_{jn} + \lambda a_{in})C_{jn} \\ =& a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} + \lambda(a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}) \\ =& |A| + 0 \end{aligned}$$

$\rightsquigarrow$ there is no change in $|A|$

Example

$$|A| = \begin{vmatrix} 1 & 2 & -1 & 4 \\ -1 & 3 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 4 & 1 & 3 \end{vmatrix} \overset{RO3s}{=} \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -1 & 6 \\ 0 & -3 & 3 & -6 \\ 0 & 2 & 2 & -1 \end{vmatrix} \overset{\alpha R_3}{=} -3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{vmatrix}$$

$$\overset{RO2}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & -1 & 6 \\ 0 & 2 & 2 & -1 \end{vmatrix} \overset{RO3s}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 4 & -5 \end{vmatrix} \overset{RO3s}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 4 & -5 \end{vmatrix}$$

$$\overset{RO3s}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 3(1 \times 1 \times 4 \times (-1)) = -12$$

Determinant of a Product

Theorem

If A and B are $n \times n$ matrices, then $|AB| = |A||B|$

Proof:

- Let E_1 be an elementary matrix that multiplies a row by a non-zero constant λ
- $|E_1| = |E_1 I| = k|I| = k$ and $|E_1 B| = k|B| = |E_1||B|$
- similarly: $|E_2 B| = -|B| = |E_2||B|$ and $|E_3 B| = |B| = |E_3||B|$
- by row equivalence we have

  $$A = E_r E_{r-1} \cdots E_1 R$$

  where R is in RREF. Since A is square, R is either I or has a row of zeros.
- $|A| = |E_r E_{r-1} \cdots E_1 R| = |E_r||E_{r-1}| \cdots |E_1||R|$ and $|E_i| \neq 0$
- If $R = I$:

  $$\begin{aligned} |AB| &= |(E_r E_{r-1} \cdots E_1 I)B| = |E_r E_{r-1} \cdots E_1 B| \\ &= |E_r||E_{r-1}|| \cdots ||E_1||B| = |E_r E_{r-1} \cdots E_1||B| = |A||B| \end{aligned}$$

- If $R \neq I$ then $|AB| = 0 = 0|B|$

Matrix Inverse using Cofactors

Theorem

If A is an $n \times n$ matrix, then A is invertible if and only if $|A| \neq 0$.

Proof:

- implied by the first theorem of today: by (4) either R is I or it has a row of zeros.
- Note also that if A is invertible then $|AA^{-1}| = |A||A^{-1}| = |I|$. Hence $|A| \neq 0$ and

$$|A^{-1}| = \frac{1}{|A|}$$

- if $|A| \neq 0$ then A is invertible: we show this by construction:

Definition (Adjoint)

If A is an $n \times n$ matrix, the matrix of cofactors of A if the matrix whose (i,j) entry is C_{ij}, the (i,j) cofactor of A.
The adjoint or (adjugate) of A is the transpose of the matrix of cofactors, ie:

$$\mathrm{adj(A)} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- $$A\,\text{adj(A)} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- entry $(1,1)$ is $a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$, ie, cofactor by row 1
  entry $(1,2)$ is $a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n}$, ie, entries of row 1 multiplied by cofactors of row 2

$$A\,\text{adj(A)} = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix} = |A|I$$

- Since $|A| \neq 0$ we can divide:

$$A\left(\frac{1}{|A|}\,\text{adj}(A)\right) = I \qquad A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$$

□

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Matrix Inverse using Cofactors

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

What is A^{-1}?

- $|A| = 1(2-1) - 2(-1-4) + 3(-1-8) = -16 \neq 0 \implies$ invertible
- Matrix of cofactors

$$\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} & -M_{14} & \cdots \\ -M_{21} & +M_{22} & -M_{23} & +M_{24} & \cdots \\ +M_{31} & -M_{32} & +M_{33} & -M_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -9 \\ 1 & -11 & 7 \\ -4 & 4 & 4 \end{bmatrix}$$

- $$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = -\frac{1}{16}\begin{bmatrix} 1 & 5 & -9 \\ 1 & -11 & 7 \\ -4 & 4 & 4 \end{bmatrix}^T = -\frac{1}{16}\begin{bmatrix} 1 & 1 & -4 \\ 5 & -11 & 4 \\ -9 & 7 & 4 \end{bmatrix}$$

Matrix Inverse using Cofactors

Example (cntd)

- Verify $AA^{-1} = I$:

$$-\frac{1}{16}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -4 \\ 5 & -11 & 4 \\ -9 & 7 & 4 \end{bmatrix} = -\frac{1}{16}\begin{bmatrix} -16 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & -16 \end{bmatrix} = I$$

Cramer's rule

Theorem (Cramer's rule)

If A is $n \times n$, $|A| \neq 0$, and $\mathbf{b} \in \mathbb{R}^n$, then the solution $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ of the linear system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{|A_i|}{|A|},$$

where A_i is the matrix obtained from A by replacing the ith column with the vector $\mathbf{b}$.

Proof: Since $|A| \neq 0$, A^{-1} exists and we can solve for $\mathbf{x}$ by multiplying $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}. The $\mathbf{x} = A^{-1}\mathbf{b}$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\Longrightarrow x_i = \frac{1}{|A|}(b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni})$, ie, cofactor expansion of column i of A with column i replaced by $\mathbf{b}$, ie, $|A_i|$ □

Matrix Inverse using Cofactors

Example

Use Cramer's rule to solve:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 7 \\ -x & + & 2y & + & z & = & -3 \\ 4x & + & y & + & z & = & 5 \end{array}$$

- In matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$$

- $|A| = -16 \neq 0$
- $$x = \frac{\begin{vmatrix} 7 & 2 & 3 \\ -3 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}}{|A|} = 1, \quad y = \frac{\begin{vmatrix} 1 & 7 & 3 \\ -1 & -3 & 1 \\ 4 & 5 & 1 \end{vmatrix}}{|A|} = -3, \quad z = \frac{\begin{vmatrix} 1 & 2 & 7 \\ -1 & 2 & -3 \\ 4 & 1 & 5 \end{vmatrix}}{|A|} = 4$$

Summary (1/2)

- There are three methods to solve $A\mathbf{x} = \mathbf{b}$ if A is $n \times n$ and $|A| \neq 0$:
    1. Gaussian elimination
    2. Matrix solution: find A^{-1}, then calculate $\mathbf{x} = A^{-1}\mathbf{b}$
    3. Cramer's rule
- There is one method to solve $A\mathbf{x} = \mathbf{b}$ if A is $m \times n$ and $m \neq n$ or if $|A| = 0$:
    1. Gaussian elimination
- There are two methods to find A^{-1}:
    1. using cofactors for the adjoint matrix
    2. by row reduction of $[A \mid I]$ to $[I \mid A^{-1}]$

Summary (2/2)

- If A is an $n \times n$ matrix, then the following statements are equivalent:
  1. A is invertible
  2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}$
  3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$
  4. the reduced row echelon form of A is I.
  5. $|A| \neq 0$

- Solving $A\mathbf{x} = \mathbf{b}$ in practice and at the computer:
  - via LU factorization (much quicker if one has to solve several systems with the same matrix A but different vectors b)
  - if A is symmetric positive definite matrix then Cholesky decomposition (twice as fast)
  - if A is large or sparse then iterative methods