DM841

Discrete Optimization

Part II

Lecture 6

Notions of Local Consistency

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Outline

1. Definitions

2. Local Consistency

3. Arc Consistency Algorithms

Reasoning with Constraints

Constraint Propagation, aka:

- constraint relaxation
- filtering algorithms
- narrowing algorithms
- constraint inference
- simplification algorithms
- label inference
- local consistency enforcing
- rules iteration
- proof rules

Local Consistency define properties that the constraint problem must satisfy *after* constraint propagation

Rules Iteration defines properties on the process of propagation itself, that is, kind and order of operations of reduction applied to the problem

Notation and Terminology

Finite domains $\rightsquigarrow$ w.l.g. $D \subseteq \mathbf{Z}$

Constraint C: relation on a (ordered) *subsequence* of variables

- $X(C) = (x_{i_1}, \dots, x_{i_{|X(C)|}})$ is the scheme or scope
- $|X(C)|$ is the arity of C (unary/binary/non-binary)
- $C \subseteq \mathbf{Z}^{|X(C)|}$ containing combinations of valid values (or tuples) $\tau \in \mathbf{Z}^{|X(C)|}$
- constraint check: testing whether a τ satisfies C
- $\mathcal{C}$: a t-tuple of constraints $\mathcal{C} = (C_1, \dots, C_t)$
- expression
  - extensional: specifies satisfying tuples (aka `table` or `extensional` via **`DFA`** or **`TupleSet`** in gecode).
    eg. $c(x_1, x_2) = \{(2,2), (2,3), (3,2), (3,3)\}$
  - intensional: specifies the characteristic function. eg. `alldiff`(x_1, x_2, x_3)

CSP

Input:

- **Variables** $X = (x_1, \ldots, x_n)$
- **Domain Expression** $\mathcal{DE} = \{x_1 \in D(x_1), \ldots, x_n \in D(x_n)\}$

a constrained satisfaction problem (CSP) is

$$\mathcal{P} = \langle X, \mathcal{DE}, \mathcal{C} \rangle$$

$\mathcal{C}$ finite set of constraints each on a subsequence of X.
$C \in \mathcal{C}$ on $Y = (y_1, \ldots, y_k)$ is $C \subseteq D(y_1) \times \ldots \times D(y_k)$

$(v_1, \ldots, v_n) \in D(x_1) \times \ldots \times D(x_n)$ is a solution of $\mathcal{P}$
if for each constraint $C_i \in \mathcal{C}$ on $x_{i_1} \ldots, x_{i_m}$ it is

$$(v_{i_1}, \ldots, v_{i_m}) \in C_i$$

CSP normalized: iff two different constraints do not involve exactly the same vars
CSP binary iff for all $C_i \in \mathcal{C}, |X(C)| = 2$

Notation and Terminology

Given a tuple τ on a sequence Y of variables and $W \subseteq Y$,

- $\tau[W]$ is the restriction of τ to variables in W (ordered accordingly)
- $\tau[x_i]$ is the value of x_i in τ
- if $X(C) = X(C')$ and $C \subseteq C'$ then for all $\tau \in C$ the reordering of τ according to $X(C')$ satisfies C'.

Example

$C(x_1, x_2, x_3): \quad x_1 + x_2 = x_3$
$C'(x_1, x_2, x_3): \quad x_1 + x_2 \leq x_3$

$C \subseteq C'$

Notation and Terminology

- Given $Y \subseteq X(C)$, $\pi_Y(C)$ denotes the projection of C on Y. It contains tuples on Y that can be extended to a tuple on $X(C)$ satisfying C.
- given $X(C_1) = X(C_2)$, the intersection $C_1 \cap C_2$ contains the tuples τ that satisfy both C_1 and C_2
- join of $\{C_1 \ldots C_k\}$ is the relation with scheme $\cup_{i=1}^{k} X(C_i)$ that contains tuples such that $\tau[X(C_i)] \in C_i$ for all $1 \leq i \leq k$.

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_i) = \{1..5\}, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv \texttt{alldiff}(x_1, x_2, x_3), C_2 \equiv x_1 \leq x_2 \leq x_3, C_3 \equiv x_4 \geq 2x_2\}\rangle$$

$\pi_{x_1,x_2}(C_1) \equiv (x_1 \neq x_2)$
$C_1 \cap C_2 \equiv (x_1 < x_2 < x_3)$
join $\{C_1, \ldots, C_3\} \equiv (x_1 < x_2 < x_3 \wedge x_4 \geq 2x_2)$

Notation and Terminology

Given $\mathcal{P} = \langle X, \mathcal{DE}, \mathcal{C} \rangle$ the instantiation I is a tuple on $Y = (x_1, \ldots, x_k) \subseteq X$: $((x_1, v_1), \ldots, (x_k, v_k))$

- I on Y is valid iff $\forall x_i \in Y$, $I[x_i] \in D(x_i)$
- I on Y is locally consistent iff it is valid and for all $C \in \mathcal{C}$ with $X(C) \subseteq Y$, $I[X(C)]$ satisfies C (some constraints may have $X(C) \not\subseteq Y$)
- a solution to $\mathcal{P}$ is an instantiation I on $X(\mathcal{C})$ which is locally consistent
- I on Y is globally consistent if it can be extended to a solution, i.e., there exists $s \in sol(\mathcal{P})$ with $I = s[Y]$

Example

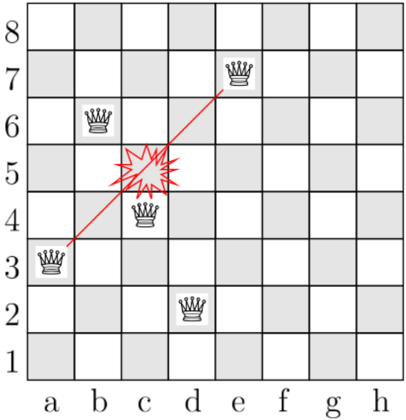
$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_i) = \{1..5\}, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv \texttt{alldiff}(x_1, x_2, x_3), C_2 \equiv x_1 \leq x_2 \leq x_3, C_3 \equiv x_4 \geq 2x_2\}\rangle$$

$\pi_{\{x_1, x_2\}}(C_1) \equiv (x_1 \neq x_2)$
$I_1 = ((x_1, 1), (x_2, 2), (x_4, 7))$ is not valid
$I_2 = ((x_1, 1), (x_2, 1), (x_4, 3))$ is local consistent since C_3 only one with $X(C_3) \subseteq Y$ and $I_2[X(C_3)]$ satisfies C_3
I_2 is not global consistent: $sol(\mathcal{P}) = \{(1, 2, 3, 4), (1, 2, 3, 5)\}$

Notation and Terminology

- An instantiation I on $\mathcal{P}$ is globally inconsistent if it cannot be extended to a solution of $\mathcal{P}$, globally consistent otherwise.
- A globally inconsistent instantiation is also called a (standard) nogood. (a partial instantiation that does not lead to a solution.)
- Remark: A locally inconsistent instantiation is a nogood. The reverse is not necessarily true

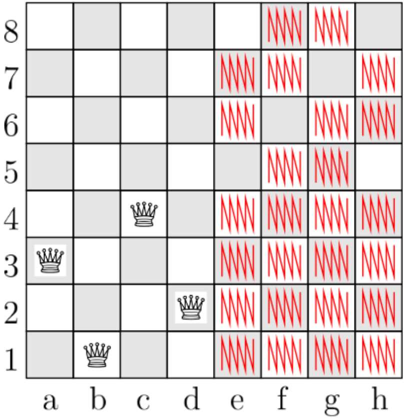
Example



$\{(x_a,3),(x_b,6),(x_c,4),(x_d,2),(x_e,7)\}$ is locally inconsistent

☞ this is a nogood.

Example



$\{(x_a,3),(x_b,1),(x_c,4),(x_d,2)\}$ is globally inconsistent

☞ this is a nogood.

Notation and Terminology

CSP solved by extending partial instantiations to global consistent ones and backtracking at local inconsitencies $\rightsquigarrow$ is NP-complete!

Idea: make the problem more explicit (tighter)

$\mathcal{P}$' is a tightening of $\mathcal{P}$ if
$X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{DE}_{\mathcal{P}'} \subseteq \mathcal{DE}, \quad \forall C \in \mathcal{C}, \exists C' \in \mathcal{C}', X(C') = X(C)$ and $C' \subseteq C$.
It implies that, any instantiation I on $Y \subseteq X_{\mathcal{P}}$ locally inconsistent for $\mathcal{P}$ is locally inconsistent for $\mathcal{P}'$.

Example

$$
\begin{aligned}
\mathcal{P} =& \langle X = (x_1, x_2, x_3), \mathcal{DE} = \{D(x_i) = [1..4], \forall i\}, \\
& \mathcal{C} = \{C_1 \equiv x_1 < x_2, C_2 \equiv x_2 < x_3, \\
& C_3 \equiv \{(111), (123), (222), (333), (234)\}\}\rangle \\
\mathcal{P}' =& \langle X, \mathcal{DE}, \mathcal{C}'\rangle, \mathcal{C}' = \{C_1, C_2, C_3' \equiv \{(123)\}\}\rangle
\end{aligned}
$$

$\mathcal{P}'$ is a tightening of $\mathcal{P}$: $X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{DE}_{\mathcal{P}'} = \mathcal{DE}$ and
$C_1 = C_1', C_2 = C_2', X(C_3) = X(C_3'), C_3' \subset C_3$. All locally inconsitent instantiations on $Y \subseteq X_{\mathcal{P}}$ for $\mathcal{P}$ are locally inconsistent for $\mathcal{P}'$. However not all solutions are preserved.

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3), \mathcal{DE} = \{D(x_i) = [1..4], \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv x_1 < x_2, C_2 \equiv x_2 < x_3, C_3 \equiv \{(111), (123), (222), (333)\}\}\rangle$$

$$\mathcal{P}' = \langle X, \mathcal{DE}, \mathcal{C}'\rangle, \mathcal{C}' = \{C_1, C_2, C_3' \equiv \{(123), (231), (312)\}\}$$

For any tuple τ on $X(C)$ that does not satisfy C there exists a constraint C' in $\mathcal{C}'$ with $X(C') \subseteq X(C)$ such that $\tau[X(C')] \notin C'$ (τ local inconsistent).
Hence $\mathcal{P}' \preceq \mathcal{P}$. But also $\mathcal{P} \preceq \mathcal{P}'$.
$\mathcal{P}'$ is not a tightening of $\mathcal{P}$: $C_3' \nsubseteq$ of any $C \in \mathcal{C}$
They are no-good equivalent.

$\rightsquigarrow$ A tightening does not define an order.

Notation and Terminology

$\mathcal{S}_{\mathcal{P}}$ is the space of all tightening for $\mathcal{P}$

We are interested in the tightenings that preserve the set of solutions ($\mathrm{sol}(\mathcal{P}') = \mathrm{sol}(\mathcal{P})$) whose space is denoted $\mathcal{S}^{\mathrm{sol}}_{\mathcal{P}}$ and among them the smallest

$\mathcal{P}^* \in \mathcal{S}^{\mathrm{sol}}_{\mathcal{P}}$ is global consistent if any instantiation I on $Y \subseteq X$ which is locally consistent in $\mathcal{P}^*$ can be extended to a solution of $\mathcal{P}$.

Computing $\mathcal{P}^*$ is exponential in time and space $\rightsquigarrow$ search a close $\mathcal{P}$ in polynomial time and space $\rightsquigarrow$ constraint propagation

- Define a property Φ that states necessary conditions on instantiations that enter in the definition of local consistency
- Reduction rules: sufficient conditions to rule out values (or instantiations) that will not be part of a solution (defined through a consistency property Φ)
  Rules iteration: set of reduction rules for each constraint that tighten the problem

Constraint Propagation

In general, we reach a $\mathcal{P}$' that is Φ consistent by constraint propagation:

- tighten $\mathcal{DE}$
- tighten $\mathcal{C}$, ex: $x_1 + x_2 \leq x_3 \rightsquigarrow x_1 + x_2 = x_3$
- add C to $\mathcal{C}$

Focus on domain-based tightenings

Domain-based tightenings

The space $\mathcal{S}_\mathcal{P}$ of domain-based tightenings of $\mathcal{P}$ is the set of problems $\mathcal{P}' = \langle X', \mathcal{DE}', \mathcal{C}' \rangle$ such that $X_{\mathcal{P}'} = X_\mathcal{P}, \quad \mathcal{DE}_{\mathcal{P}'} \subseteq \mathcal{DE}, \quad \mathcal{C}' = \mathcal{C}$

Task:
Finding a tightening $\mathcal{P}^*$ in $\mathcal{S}_\mathcal{P}^{\mathrm{sol}} \subseteq \mathcal{S}_\mathcal{P}$ (the set that contains all problems that preserve the solutions of $\mathcal{P}$) such that:
forall $x_i \in X_\mathcal{P}$, $D_{\mathcal{P}*}(x_i)$ contains only values that belong to a solution itself, i.e., $D_{\mathcal{P}*}(x_i) = \pi_{\{x_i\}}(\mathrm{sol}(\mathcal{P}))$

It is clearly NP-hard since it corresponds to solving $\mathcal{P}$ itself.

- Reduction rules:

  $$D(x_i) \leftarrow D(x_i) \cap \{v_i | D(x_1) \times D(x_j - 1) \times \{v_i\} \times \ldots D(x_j + 1) \times \ldots D(x_k) \cap C \neq \emptyset\}$$

  (the rule is parameterised by a variable x_i and a constraint C)
  Rules iteration (for all i)

It is clearly NP-hard since it corresponds to solving $\mathcal{P}$ itself.
$\leadsto$ hence polynomial reduction rules to approximate $\mathcal{P}^*$

Apply rules iteration for each constraint. Domain-based reduction rules are also called propagators.

Example

$C = (|x_1 - x_2| = k)$
Propagator: $D(x_1) \leftarrow D(x_1) \cap [\min_D(x_2) - k .. \max_D(x_2) + k]$

Rather than defining rules we define Φ: e.g., unary, arc, path, k-consistency

Domain-based local consistency

Domain-based local consistency property Φ specifies a necessary condition on values to belong to solutions. We restrict to those stable under union.

A domain-based property Φ is stable under union iff for any Φ-consistent problem $\mathcal{P}_1 = (X, \mathcal{DE}, \mathcal{C})$ and $\mathcal{P}_2 = (X, \mathcal{DE}, \mathcal{C})$ the problem $\mathcal{P}' = (X, \mathcal{DE}_1 \cup \mathcal{DE}_2, C)$ is Φ-consistent.

Example

Φ for each constraint C and variable $x_i \in X(C)$, at least half of the values in $D(x_i)$ belong to a valid tuple satisfying C.

$$\mathcal{P}_1 = \langle X = (x_1, x_2), \mathcal{DE} = \{D_1(x_1) = \{1, 2\}, D_1(x_2) = \{2\}\}, C \equiv \{x_1 = x_2\}\rangle$$
$$\mathcal{P}_2 = \langle X = (x_1, x_2), \mathcal{DE} = \{D_2(x_1) = \{2, 3\}, D_2(x_2) = \{2\}\}, C \equiv \{x_1 = x_2\}\rangle$$

Both are Φ consistent but they are not stable under union.

Domain-based tightenings

Note: Not all Φ-consistent tightenings preserve the solutions
We search for the Φ-closure $\Phi(\mathcal{P})$ (the union of all Φ-consistent $\mathcal{P}' \in \mathcal{S}_{\mathcal{P}}$)

If Φ is stable under union, then $\Phi(\mathcal{P})$ is the unique domain-based Φ-consistent tightening problem that contains all others.

$\mathrm{sol}(\phi(\mathcal{P})) = \mathrm{sol}(\mathcal{P})$

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_i) = \{1,2\}, \forall i\}, \\ \mathcal{C} = \{C_1 \equiv x_1 \leq x_2, C_2 \equiv x_2 \leq x_3, C_3 \equiv x_1 \neq x_3\}\rangle$$

Φ all values for all variables can be extended consistently to a second variable

$$\mathcal{P}' = \langle X = (x_1, x_2, x_3, x_4), \mathcal{DE} = \{D(x_1) = 1, D(x_2) = 1, D(x_3) = 2, \forall i\}, \\ \mathcal{C} = \{C_1 \equiv x_1 \leq x_2, C_2 \equiv x_2 \leq x_3, C_3 \equiv x_1 \neq x_3\}\rangle$$

$\mathcal{P}$' is consistent but it does not contain $(1,2,2)$ which is in $\mathrm{sol}(\mathcal{P})$
$\Phi(\mathcal{P}) : \langle X, \mathcal{DE}_\Phi, \mathcal{C}\rangle$ with $D_\Phi(x_1) = 1, D_\Phi(x_2) = \{1,2\}, D_\Phi(x_3) = 2$

Definition

A set has closure under an operation if performance of that operation on members of the set always produces a member of the same set.

A set is said to be closed under a collection of operations if it is closed under each of the operations individually.

Domain-based tightenings

Proposition (Fixed Point): If a domain based consistency property Φ is stable under union, then for any $\mathcal{P}$, the $\mathcal{P}$' with $\mathcal{DE}_{\mathcal{P}'}$ obtained by iteratively removing values that do not satisfy Φ until no such value exists is the Φ-closure of $\mathcal{P}$.

Contrary to $\mathcal{P}^*$, $\Phi(\mathcal{P})$ can be computed by a greedy algorithm:

Corollary If a domain-based consistency property Φ is polynomial to check, finding $\Phi(\mathcal{P})$ is polynomial as well.

enforcing Φ consistency $\equiv$ finding closure $\Phi(\mathcal{P})$