DM554/DM545

Linear and Integer Programming

Lecture 11
Relaxations
Well Solved Problems
Network Flows

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

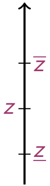
Outline

Outline

Optimality and Relaxation

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

How can we prove that $\mathbf{x}^*$ is optimal?
$\overline{z}$ is UB
$\underline{z}$ is LB
stop when $\overline{z} - \underline{z} \leq \epsilon$



- Primal bounds (here lower bounds): every feasible solution gives a primal bound
  may be easy or hard to find, heuristics
- Dual bounds (here upper bounds): Relaxations

Optimality gap:

$$\text{gap} = \frac{db - pb}{\sup\{|z|, z \in [pb, db]\}} (\cdot 100) \qquad \text{for a maximization problem}$$

(If $pb \geq 0$ and $db \geq 0$ then $\frac{db-pb}{db}$. If $db = pb = 0$ then $\text{gap} = 0$. If no feasible sol found or $pb \leq 0 \leq db$ then gap is not computed.)

Proposition

$$\begin{aligned} (RP)\ z^R &= \max\{f(\mathbf{x}) : \mathbf{x} \in T \subseteq \mathbb{R}^n\} \textit{ is a relaxation of} \\ (IP)\ z\ \ &= \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{R}^n\} \textit{ if :} \end{aligned}$$

(i) $X \subseteq T$ *or*

(ii) $f(\mathbf{x}) \geq c(\mathbf{x})\ \forall \mathbf{x} \in X$

In other terms:

$$\max_{\mathbf{x} \in T} f(\mathbf{x}) \geq \begin{Bmatrix} \max_{\mathbf{x} \in T} c(\mathbf{x}) \\ \max_{\mathbf{x} \in X} f(\mathbf{x}) \end{Bmatrix} \geq \max_{\mathbf{x} \in X} c(\mathbf{x})$$

- T: candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(\mathbf{x}) \geq c(\mathbf{x})$

Relaxations

How to construct relaxations?

1. $IP : \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P \cap \mathbb{Z}^n\}, P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$
   $LP : \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P\}$
   Better formulations give better bounds ($P_1 \subseteq P_2$)

Proposition

(i) *If a relaxation RP is infeasible, the original problem IP is infeasible.*

(ii) *Let $\mathbf{x}^*$ be optimal solution for RP. If $\mathbf{x}^* \in X$ and $f(\mathbf{x}^*) = c(\mathbf{x}^*)$ then $\mathbf{x}^*$ is optimal for IP.*

2. Combinatorial relaxations to easy problems that can be solved rapidly
   Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$IP: \quad z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

$$LR: \quad z(\mathbf{u}) = \max\{\mathbf{c}^T\mathbf{x} + \mathbf{u}(\mathbf{b} - A\mathbf{x}) : \mathbf{x} \in X\}$$

$$z(\mathbf{u}) \geq z \qquad \forall \mathbf{u} \geq \mathbf{0}$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X\} \qquad w = \min\{w(\mathbf{u}) : \mathbf{u} \in U\}$$

form a weak-dual pair if $c(\mathbf{x}) \leq w(\mathbf{u})$ for all $\mathbf{x} \in X$ and all $\mathbf{u} \in U$.
When $z = w$ they form a strong-dual pair

Proposition

$z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}_+^n\}$ and $w^{LP} = \min\{\mathbf{u}^T\mathbf{b} : A^T\mathbf{u} \geq \mathbf{c}, \mathbf{u} \in \mathbb{R}_+^m\}$ (ie, dual of linear relaxation) form a weak-dual pair.

Proposition

Let IP and D be weak-dual pair:

(i) *If D is unbounded, then IP is infeasible*

(ii) *If $\mathbf{x}^* \in X$ and $\mathbf{u}^* \in U$ satisfy $c(\mathbf{x}^*) = w(\mathbf{u}^*)$ then $\mathbf{x}^*$ is optimal for IP and $\mathbf{u}^*$ is optimal for D.*

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Examples

Weak pairs:
Matching: $z = \max\{\mathbf{1}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^m\}$
V. Covering: $w = \min\{\mathbf{1}^T\mathbf{y} : A^T y \geq \mathbf{1}, \mathbf{y} \in \mathbb{Z}_+^n\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:
Packing: $z = \max\{\mathbf{1}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^n\}$
S. Covering: $w = \min\{\mathbf{1}^T\mathbf{y} : A^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \in \mathbb{Z}_+^m\}$

Outline

1. Relaxations

2. Well Solved Problems

Separation problem

$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\} \equiv \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \text{conv}(X)\}$
$X \subseteq \mathbb{Z}^n$, P a polyhedron $P \subseteq \mathbb{R}^n$ and $X = P \cap \mathbb{Z}^n$

Definition (Separation problem for a COP)

Given $\mathbf{x}^* \in P$ is $\mathbf{x}^* \in \text{conv}(X)$? If not find an inequality $\mathbf{ax} \leq \mathbf{b}$ satisfied by all points in X but violated by the point $\mathbf{x}^*$.

(Farkas' lemma states the existence of such an inequality.)

Properties of Easy Problems

Four properties that often go together:

Definition

(i) Efficient optimization property: $\exists$ a polynomial algorithm for $\max\{\mathbf{cx} : \mathbf{x} \in X \subseteq \mathbb{R}^n\}$

(ii) Strong duality property: $\exists$ strong dual D $\min\{w(\mathbf{u}) : \mathbf{u} \in U\}$ that allows to quickly verify optimality

(iii) Efficient separation problem: $\exists$ efficient algorithm for separation problem

(iv) Efficient convex hull property: a compact description of the convex hull is available

Example:

If explicit convex hull   strong duality holds
efficient separation property (just description of $\text{conv}(X)$)

Theoretical analysis to prove results about

- strength of certain inequalities that are facet defining
  2 ways
- descriptions of convex hull of some discrete $X \subseteq \mathbb{Z}^*$
  several ways, we see one next

Example

Let

$$X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^{m} x_i \leq my, x_i \leq 1 \text{ for } i = 1, \ldots, m\}$$

$$P = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$$

.

Polyhedron P describes $\text{conv}(X)$

Totally Unimodular Matrices

When the LP solution to this problem

$$IP : \max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n_+\}$$

with all data integer will have integer solution?

$$\left[\begin{array}{c|c|c|c} A_N & A_B & \mathbf{0} & \mathbf{b} \\ \hline \mathbf{c}_N^T & \mathbf{c}_B^T & 1 & 0 \end{array}\right]$$

$A_B x_B + A_N x_N = b$

$\mathbf{x}_N = \mathbf{0} \rightsquigarrow A_B \mathbf{x}_B = \mathbf{b}$,
A_B $m \times m$ non singular matrix
$\mathbf{x}_B \geq 0$

Cramer's rule for solving systems of linear equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \mathbf{x} = A_B^{-1}\mathbf{b} = \frac{A_B^{adj}\mathbf{b}}{\det(A_B)}$$

Definition

- A square integer matrix B is called unimodular (UM) if $\det(B) = \pm 1$
- An integer matrix A is called totally unimodular (TUM) if every square, nonsingular submatrix of A is UM

Proposition

- *If A is TUM then all vertices of $R_1(A) = \{x : Ax = b, x \geq 0\}$ are integer if b is integer*
- *If A is TUM then all vertices of $R_2(A) = \{x : Ax \leq b, x \geq 0\}$ are integer if b is integer.*

Proof: if A is TUM then $\left[A \mid I \right]$ is TUM
Any square, nonsingular submatrix C of $\left[A \mid I \right]$ can be written as

$$C = \left[\begin{array}{c|c} B & 0 \\ \hline D & I_k \end{array} \right]$$

where B is square submatrix of A. Hence $\det(C) = \det(B) = \pm 1$

Proposition

The transpose matrix A^T of a TUM matrix A is also TUM.

Theorem (Sufficient condition)

An integer matrix A with is TUM if

1. $a_{ij} \in \{0, -1, +1\}$ *for all* i, j
2. *each column contains at most two non-zero coefficients* ($\sum_{i=1}^{m} |a_{ij}| \leq 2$)
3. *if the rows can be partitioned into two sets* I_1, I_2 *such that:*
   - *if a column has 2 entries of same sign, their rows are in different sets*
   - *if a column has 2 entries of different signs, their rows are in the same set*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: by induction

Basis: one matrix of one element $\{+1, -1\}$ is TUM

Induction: let C be of size k.
If C has column with all 0s then it is singular.
If a column with only one 1 then expand on that by induction
If 2 non-zero in each column then

$$\forall j : \sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij}$$

but then linear combination of rows and $\det(C) = 0$

Other matrices with integrality property:

- TUM
- Balanced matrices
- Perfect matrices
- Integer vertices

Defined in terms of forbidden substructures that represent fractionating possibilities.

Proposition

A is always TUM if it comes from

- *node-edge incidence matrix of undirected bipartite graphs (ie, no odd cycles)* ($I_1 = U, I_2 = V, B = (U, V, E)$)
- *node-arc incidence matrix of directed graphs* ($I_2 = \emptyset$)

Eg: Shortest path, max flow, min cost flow, bipartite weighted matching

Summary

1. Relaxations

2. Well Solved Problems