DM559

Linear and Integer Programming

Lecture 3

Matrices and Vectors: Geometric Insight

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Outline

1. Geometric Insight

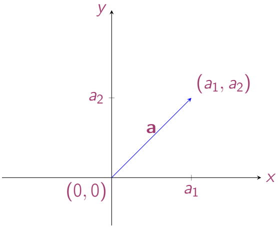
2. Linear Systems

Outline

Geometric Insight

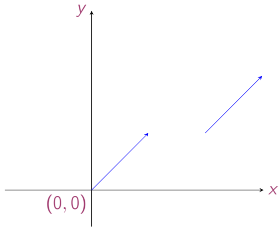
- Set $\mathbb{R}$ can be represented by real-number line. Set $\mathbb{R}^2$ of real number pairs (a_1, a_2) can be represented by the Cartesian plane.

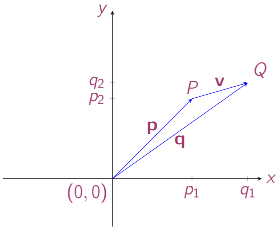
- To a point in the plane $A = (a_1, a_2)$ it is associated a position vector $\mathbf{a} = (a_1, a_2)^T$, representing the displacement from the origin $(0, 0)$. ◇



- Two displacement vectors of same length and direction are considered to be equal even if they do not both start from the origin

- If object displaced from O to P by displacement $\mathbf{p}$ and from P to Q by displacement $\mathbf{v}$, then the total displacement satisfies $\mathbf{q} = \mathbf{p} + \mathbf{v} = \mathbf{v} + \mathbf{q}$





- $\mathbf{v} = \mathbf{q} - \mathbf{p}$, think of $\mathbf{v}$ as the vector that is added to $\mathbf{p}$ to obtain $\mathbf{q}$.

- the length of a vector $\mathbf{a} = (a_1, a_2)^T$ is denoted by $||\mathbf{a}||$ and from Pythagoras

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2} = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$$

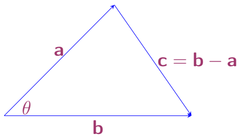
- the direction is given by the components of the vector
- the unit vector can be derived from

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

Theorem (Inner Product)

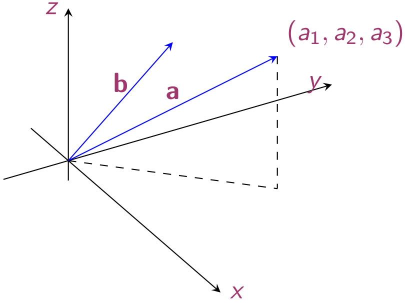
Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and let θ denote the angle between them. Then,

$$\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$



Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (or normal or perpendicular) if and only if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$.

Vectors in $\mathbb{R}^3$



$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Lines in $\mathbb{R}^2$

- Cartesian line equation $y = ax + b$

- another way is by giving position vectors.
  We can let $x = t$ where t is any real number. Then
  $y = ax + b = at + b$. Hence the position vector $\mathbf{x} = (x, y)^T$

$$\mathbf{x} = \begin{bmatrix} t \\ at + b \end{bmatrix} = t \begin{bmatrix} 1 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = t\mathbf{v} + (0, b)^T, \qquad t \in \mathbb{R}$$

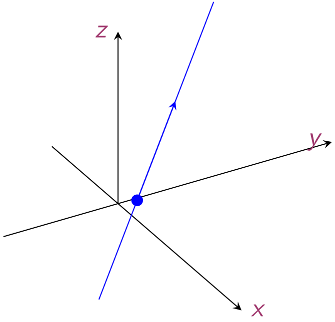
- To derive the Cartesian equation: locate one particular point on the line, eg, the y intercept. Then the position vector of any point on the line is a sum of two displacements, first going to the point and then along the direction of the line. Try with $P = (-1, 1)$ and $Q = (3, 2)$

- In general, any line in $\mathbb{R}^2$ is given by a vector equation with one parameter of the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

  where $\mathbf{x}$ is the position vector, $\mathbf{p}$ is any particular point and $\mathbf{v}$ is the direction of the line

Lines in $\mathbb{R}^3$



$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} + s \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Are these lines intersecting?
What is the Cartesian equation of the first?

In $\mathbb{R}^2$, two lines are:

- parallel
- intersecting in a unique point

In $\mathbb{R}^3$, two lines are:

- parallel
- intersecting in a unique point
- skew (lay on two parallel planes)

What about these lines? Do they intersect? Are they coplanar?

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

Planes in $\mathbb{R}^3$

Vector parametric equation:

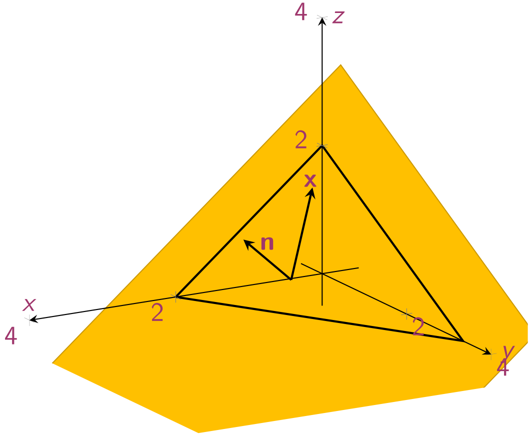
- The position of vectors of points on a plane is described by:

  $$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

  provided $\mathbf{v}$ and $\mathbf{w}$ are non-zero and not parallel.
  ($\mathbf{p}$ position vector, $\mathbf{v}$ and $\mathbf{w}$ displacement vectors).

- How is the plane through the origin? What if $\mathbf{v}$ and $\mathbf{w}$ are parallel?

- Two intersecting lines determine a plane. What is its description?


4
z
2
x
n
x
2
4
2
y
4

Alternative Description of Planes

Cartesian equation:

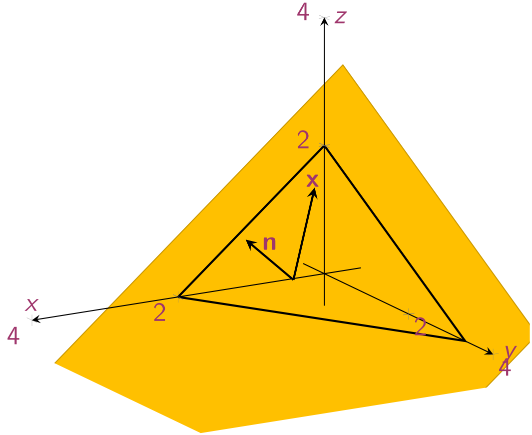
- Let $\mathbf{n}$ be a given vector in $\mathbb{R}^3$. All positions represented by postion vectors $\mathbf{x}$ that are orthogonal to $\mathbf{n}$ describe a plane through the origin. ($\mathbf{n}$ is called a normal vector to the plane)

- Vectors $\mathbf{n}$ and $\mathbf{x}$ are orthogonal iff

  $$\langle \mathbf{n}, \mathbf{x} \rangle = 0,$$

  hence this equation describes a plane.
  If $\mathbf{n} = (a, b, c)^T$ and $\mathbf{x} = (x, y, z)^T$, then the equation becomes:

  $$ax + by + cz = 0$$


4
z
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4

- For a point P on the plane with position vector $\mathbf{p}$ and a position vector $\mathbf{x}$ of any other point on the plane, the displacement vector $\mathbf{x} - \mathbf{p}$ lies on the plane and $\mathbf{n} \perp \mathbf{x} - \mathbf{p}$

- Conversely, if the position vector $\mathbf{x}$ of a point is such that

$$\langle \mathbf{n}, \mathbf{x} - \mathbf{p} \rangle = 0$$

  then the point represented by $\mathbf{x}$ lies on the plane.

- hence, $\langle \mathbf{n}, \mathbf{x} \rangle = \langle \mathbf{n}, \mathbf{p} \rangle = d$ and the equation becomes:

$$ax + by + cz = d$$

Eg.: $2x - 3y - 5z = 2$ has $\mathbf{n} = (2, -3, -5)^T$ and passes through $(0, 0, e)$

Vector parametric equation $\Longleftrightarrow$ Cartesian equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} = s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

$$3x - y + z = 0, \qquad \mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\langle \mathbf{n}, \mathbf{v} \rangle = 0, \langle \mathbf{n}, \mathbf{w} \rangle = 0 \text{ and } \langle \mathbf{n}, s\mathbf{v} + t\mathbf{w} \rangle = 0 \text{ for } s, t \in \mathbb{R}$$

What changes if the plane does not pass through the origin?

Are the two following planes parallel?

$$x + 2y - 3x = 0 \text{ and } -2x - 4y + 6z = 4$$

and these?

$$x + 2y - 3x = 0 \text{ and } x - 2y + 5z = 4$$

Lines and Hyperplanes in $\mathbb{R}^n$

- Point in $\mathbb{R}^n$: $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$
- Length of a vector $\mathbf{x} = (x_, x_2, \ldots, x_n)^T$

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

- The vectors in $\mathbb{R}^n$ are orthogonal iff

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0.$$

- Line:

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbb{R}$$

How many Cartesian equations?

- The set of points $(x_1, x_2, \ldots, x_n)$ that satisfy a Cartesian equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$$

is called hyperplane. ($\langle \mathbf{n}, \mathbf{x} - \mathbf{p} \rangle = 0$.) What is the vector equation?

Outline

Systems of Linear Equations

Definition (System of linear equations, aka linear system)

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of m equations of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

The numbers a_{ij} are known as the coefficients of the system.

We say that $s_1, s_2, \ldots, s_n$ is a solution of the system if all m equations hold true when

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

Examples

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 4 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & + & x_5 & = & 5 \\ x_1 & & & & & + & x_4 & + & x_5 & = & 4 \end{array}$$

has solution

$$x_1 = -1, x_2 = -2, x_3 = 1, x_4 = 3, x_5 = 2.$$

Is it the only one?

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 4 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & + & x_5 & = & 5 \\ x_1 & & & & & + & x_4 & + & x_5 & = & 6 \end{array}$$

has no solutions

Definition (Coefficient Matrix)

The matrix $A = (a_{ij})$, whose (i, j) entry is the coefficient a_{ij} of the system of linear equations is called the coefficient matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ then

$$\underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{n \times 1}{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}}$$

hence, the linear system can be written also as $A\mathbf{x} = \mathbf{b}$

Row operations

How do we find solutions?

$$\begin{array}{l|rcrcrcl}
\texttt{R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\
\texttt{R2:} & 2x_1 & + & x_2 & + & x_3 & = & 4 \\
\texttt{R3:} & x_1 & - & x_2 & + & 2x_3 & = & 5
\end{array}$$

Eliminate one of the variables from two of the equations

$$\begin{array}{l|rcrcrcr}
\texttt{R1'=R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\
\texttt{R2'=R2-2*R1:} & & - & x_2 & - & x_3 & = & -2 \\
\texttt{R3'=R3:} & x_1 & - & x_2 & + & 2x_3 & = & 5
\end{array}$$

$$\begin{array}{l|rcrcrcr}
\texttt{R1'=R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\
\texttt{R2'=R2:} & & - & x_2 & - & x_3 & = & -2 \\
\texttt{R3'=R3-R1:} & & - & 2x_2 & + & x_3 & = & 2
\end{array}$$

We can now eliminate one of the variables in the last two equations to obtain the solution

Row operations that do not alter solutions:

O1: multiply both sides of an equation by a non-zero constant

O2: interchange two equations

O3: add a multiple of one equation to another

These operations only act on the coefficients of the system
For a system $A\mathbf{x} = \mathbf{b}$:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}$$

Augmented Matrix

Definition (Augmented Matrix and Elementary row operations)

For a system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

the augmented matrix of the system and the row operations are:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

RO1: multiply a row by a non-zero constant

RO2: interchange two rows

RO3: add a multiple of one row to another