DM559

Linear and Integer Programming

Lecture 5
Matrix Inverse and Determinants

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Outline

1. Elementary Matrices

2. Matrix Inverse

3. Determinants

4. Matrix Inverse and Cramer's rule

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1. Elementary Matrices

2. Matrix Inverse

3. Determinants

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Row Operations Revisited

Let's examine the process of applying the elementary row operations:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \overrightarrow{\mathbf{a}}_2 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix}$$

($\overrightarrow{\mathbf{a}}_i$ row ith of matrix A)
Then the three operations can be described as:

$$\begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \lambda \overrightarrow{\mathbf{a}}_2 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix} \qquad \begin{bmatrix} \overrightarrow{\mathbf{a}}_2 \\ \overrightarrow{\mathbf{a}}_1 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix} \qquad \begin{bmatrix} \overrightarrow{\mathbf{a}}_1 \\ \overrightarrow{\mathbf{a}}_2 + \lambda \overrightarrow{\mathbf{a}}_1 \\ \vdots \\ \overrightarrow{\mathbf{a}}_n \end{bmatrix}$$

For any $n \times n$ matrices A and B:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_n B \end{bmatrix}$$

$$\begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B + \lambda \vec{\mathbf{a}}_1 B \\ \vdots \\ \vec{\mathbf{a}}_n B \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ (\vec{\mathbf{a}}_2 + \lambda \vec{\mathbf{a}}_1) B \\ \vdots \\ \vec{\mathbf{a}}_n B \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 + \lambda \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_n \end{bmatrix} B$$

(matrix obtained by a row operation on AB)

$= $ (matrix obtained by a row operation on A)B

(matrix obtained by a row operation on B)

$= $ (matrix obtained by a row operation on I)B

Elementary matrix

Definition (Elementary matrix)

An elementary matrix, E, is an $n \times n$ matrix obtained by doing exactly one row operation on the $n \times n$ identity matrix, I.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{ii-i} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{ii-i} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$E_1 B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Matrix Inverse

The three elementary row operations are trivially invertible.

Theorem

Any elementary matrix is invertible, and the inverse is also an elementary matrix

$$E_1 B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1}(E_1 B) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$$

Row equivalence

To be an equivalence relation a relation must satisfy three properties:

- reflexive: $A \sim B$
- symmetric: $A \sim B \implies B \sim A$
- transitive: $A \sim B$ and $B \sim C \implies A \sim C$

Definition (Row equivalence)

If two matrices A and B are $m \times n$ matrices, we say that A is row equivalent to B if and only if there is a sequence of elementary row operations to transform A to B.

Theorem

Every matrix is row equivalent to a matrix in reduced row echelon form

Invertible Matrices

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A^{-1} *exists*
2. $A\mathbf{x} = \mathbf{b}$ *has a unique solution for any* $\mathbf{b} \in \mathbb{R}^n$
3. $A\mathbf{x} = \mathbf{0}$ *only has the trivial solution,* $\mathbf{x} = \mathbf{0}$
4. *The reduced row echelon form of* A *is* I.

Proof: $(1) \implies (2) \implies (3) \implies (4) \implies (1)$.

- $(1) \implies (2)$

  $$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \implies I\mathbf{x} = A^{-1}\mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$$

  hence $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution and it is unique, indeed:
  $A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}, \quad \forall \mathbf{b}$

- $(2) \implies (3)$
  If $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$, then this is true for $\mathbf{b} = 0$. The unique solution of $A\mathbf{x} = \mathbf{0}$ must be the trivial solution, $\mathbf{x} = \mathbf{0}$

- (3) $\implies$ (4)
  then in the reduced row echelon form of A there are no non-leading (free) variables and there is a leading one in every column hence also a leading one in every row (because A is square and in RREF) hence it can only be the identity matrix

- (4) $\implies$ (1)
  $\exists$ sequence of row operations and elementary matrices $E_1, \ldots, E_r$ that reduce A to I ie,

  $$E_r E_{r-1} \cdots E_1 A = I$$

  Each elementary matrix has an inverse hence multiplying repeatedly on the left by E_r^{-1}, E_{r-1}^{-1}:

  $$A = E_1^{-1} \cdots E_{r-1}^{-1} E_r^{-1} I$$

  hence, A is a product of invertible matrices hence invertible.
  (Recall that $B^{-1}A^{-1} = (AB)^{-1}$)

Outline

Matrix Inverse via Row Operations

We saw that:

$$A = E_1^{-1} \cdots E_{r-1}^{-1} E_r^{-1} I$$

taking the inverse of both sides:

$$A^{-1} = (E_1^{-1} \cdots E_{r-1}^{-1} E_r^{-1})^{-1} = E_r \cdots E_1 = E_r \cdots E_1 I$$

Hence:

$$\text{if} \quad E_r E_{r-1} E \cdots E_1 A = I \quad \text{then} \quad A^{-1} = E_r E_{r-1} \cdots E_1 I$$

Method:

- Construct $[A \mid I]$
- Use row operations to reduce this to $[I \mid B]$
- If this is not possible then the matrix is not invertible
- If it is possible then $B = A^{-1}$

Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \to [A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[iii+i]{ii-i} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{iii-2ii} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \xrightarrow[ii-2iii]{i-4iii} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -11 & 8 & -4 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{i-2ii} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -7 & 5 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Verify by checking $AA^{-1} = I$ and $A^{-1}A = I$.
What would happen if the matrix is not invertible?

Verifying an Inverse

Theorem

If A and B are $n \times n$ matrices and $AB = I$, then A and B are each invertible matrices, and $A = B^{-1}$ and $B = A^{-1}$.

Proof: show that $B\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = 0$, then B is invertible.

$$B\mathbf{x} = \mathbf{0} \implies A(B\mathbf{x}) = A\mathbf{0} \implies (AB)\mathbf{x} = \mathbf{0} \overset{AB=I}{\implies} I\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$$

So B^{-1} exists for the previous theorem. Hence:

$$AB = I \implies (AB)B^{-1} = IB^{-1} \implies A(BB^{-1}) = B^{-1} \implies A = B^{-1}$$

So A is the inverse of B, and therefore also invertible and

$$A^{-1} = (B^{-1})^{-1} = B$$

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Determinants

- The determinant of a matrix A is a particular number associated with A, written $|A|$ or $\det(A)$, that tells whether the matrix A is invertible.

- For the 2×2 case:

$$[A \mid I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{(1/a)R_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - cb/a & -c/a & 1 \end{array}\right] \xrightarrow{aR_2} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & (ad - bc) & -c & a \end{array}\right]$$

Hence A^{-1} exists if and only if $ad - bc \neq 0$.

- hence, for a 2×2 matrix the determinant is

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- The extension to $n \times n$ matrices is done recursively

Definition (Minor)

For an $n \times n$ matrix the (i, j) minor of A, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and the jth column of A.

Definition (Cofactor)

The (i, j) cofactor of a matrix A is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Definition (Cofactor Expansion of $|A|$ by row one)

The determinant of an $n \times n$ matrix is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 1 \\ -1 & 3 & 0 \end{bmatrix}$$

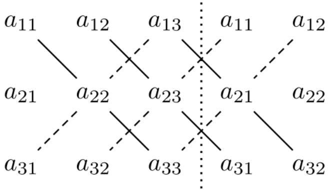
$$\begin{aligned} |A| &= 1C_{11} + 2C_{12} + 3C_{13} \\ &= 1\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ -1 & 3 \end{vmatrix} \\ &= 1(-3) - 2(1) + 3(13) = 34 \end{aligned}$$

Theorem

If A is an $n \times n$ matrix, then the determinant of A can be computed by multiplying the entries of any row (or column) by their cofactors and summing the resulting products:

$$\begin{aligned} |A| =& a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ & \textit{(cofactor expansion by row } i\textit{)} \\ |A| =& a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \\ & \textit{(cofactor expansion by column } j\textit{)} \end{aligned}$$

A mnemonic rule for the 3×3 matrix determinant: the rule of Sarrus



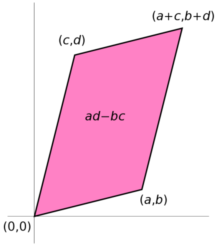
$$
\begin{aligned}
|A| = &+ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{aligned}
$$

Verify the rule:

- from the conditions of existence of an inverse
- as a consequence of the general recursive rule for the determinants

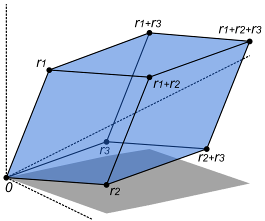
Geometric interpretation

2×2



The area of the parallelogram is the absolute value of the determinant of the matrix formed by the vectors representing the parallelogram's sides.

3×3



The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the rows constructed from the vectors $r1$, $r2$, and $r3$.

Properties of Determinants

Let A be an $n \times n$ matrix, then it follows from the previous theorem:

1. $|A^T| = |A|$
2. If a row of A consists entirely of zeros, then $|A| = 0$.
3. If A contains two rows which are equal, then $|A| = 0$.

$$|A| = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = -d \begin{vmatrix} b & c \\ b & c \end{vmatrix} + e \begin{vmatrix} a & c \\ a & c \end{vmatrix} - f \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 + 0 + 0$$

For 1. we can substitute row with column in 2., 3., 4.

4. If the cofactors of one row are multiplied by the entries of a different row and added, then the result is 0. That is, if $i \neq j$, then $a_{j1}C_{i1} + a_{j2}C_{i2} + \cdots + a_{jn}C_{in} = 0$.

$$A = \begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \; i\text{th} \qquad |A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$B = \begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \; i\text{th} \qquad |B| = a_{j1}C_{i1} + a_{j2}C_{i2} + \cdots + a_{jn}C_{in} = 0$$

5. If $A = (a_{ij})$ and if each entry of one of the rows, say row i, can be expressed as a sum of two numbers, $a_{ij} = b_{ij} + c_{ij}$ for $i \leq j \leq n$, then $|A| = |B| + |C|$, where B is the matrix A with row i replaced by $b_{i1}, b_{i2}, \cdots, b_{in}$ and C is the matrix A with row i replaced by $c_{i1}, c_{i2}, \cdots, c_{in}$.

$$|A| = \begin{vmatrix} a & b & c \\ d+p & e+q & f+r \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ g & h & i \end{vmatrix} = |B| + |C|$$

Triangular Matrices

Definition (Triangular Matrices)

An $n \times n$ matrix is said to be upper triangular if $a_{ij} = 0$ for $i > j$ and lower triangular if $a_{ij} = 0$ for $i < j$. Also A is said to be triangular if it is either upper triangular or lower triangular.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition (Diagonal Matrices)

An $n \times n$ matrix is diagonal if $a_{ij} = 0$ whenever $i \neq j$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Determinant using row operations

- Which row or column would you choose for the cofactor expansion in this case:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} =? = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

- if A is upper/lower triangular or diagonal, then $|A| = a_{11}a_{22}\cdots a_{nn}$

- Idea: a square matrix in REF is upper triangular. What is the effect of row operations on the determinant?

RO1 multiply a row by a non-zero constant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad |B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$|B| = \alpha a_{i1} C_{i1} + \alpha a_{i2} C_{i2} + \cdots + \alpha a_{in} C_{in} = \alpha |A|$$

$\rightsquigarrow$ $|A|$ changes to $\alpha|A|$

RO2 interchange two rows

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \qquad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad \implies |B| = -|A|$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad |B| = \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \implies |B| = -|A|$$

$\rightsquigarrow$ $|A|$ changes to $-|A|$ (by induction)

RO3 add a multiple of one row to another

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad |B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} + 4a_{11} & a_{22} + 4a_{12} & \cdots & a_{2n} + 4a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\begin{aligned} |B| =& (a_{j1} + \lambda a_{i1})C_{j1} + (a_{j2} + \lambda a_{i2})C_{j2} + \cdots + (a_{jn} + \lambda a_{in})C_{jn} \\ =& a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} + \lambda(a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}) \\ =& |A| + 0 \end{aligned}$$

$\rightsquigarrow$ there is no change in $|A|$

Example

$$|A| = \begin{vmatrix} 1 & 2 & -1 & 4 \\ -1 & 3 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 4 & 1 & 3 \end{vmatrix} \overset{RO3s}{=} \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -1 & 6 \\ 0 & -3 & 3 & -6 \\ 0 & 2 & 2 & -1 \end{vmatrix} \overset{\alpha R_3}{=} -3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{vmatrix}$$

$$\overset{RO2}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & -1 & 6 \\ 0 & 2 & 2 & -1 \end{vmatrix} \overset{RO3s}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 4 & -5 \end{vmatrix} \overset{RO3s}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 4 & -5 \end{vmatrix}$$

$$\overset{RO3s}{=} 3 \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 3(1 \times 1 \times 4 \times (-1)) = -12$$

Determinant of a Product

Theorem

If A and B are $n \times n$ matrices, then $|AB| = |A||B|$

<u>Proof</u>:

- Let E_1 be an elementary matrix that multiplies a row by a non-zero constant λ
- $|E_1| = |E_1 I| = k|I| = k$ and $|E_1 B| = k|B| = |E_1||B|$
- similarly: $|E_2 B| = -|B| = |E_2||B|$ and $|E_2 B| = |B| = |E_2||B|$
- by row equivalence we have

  $$A = E_r E_{r-1} \cdots E_1 R$$

  where R is in RREF. Since A is square, R is either I or has a row of zeros.
- $|A| = |E_r E_{r-1} \cdots E_1 R| = |E_r||E_{r-1}| \cdots |E_1||R|$ and $|E_i| \neq 0$
- If $R = I$:

  $$\begin{aligned} |AB| &= |(E_r E_{r-1} \cdots E_1 I)B| = |E_r E_{r-1} \cdots E_1 B| \\ &= |E_r||E_{r-1}| \cdots |E_1||B| = |E_r E_{r-1} \cdots E_1||B| = |A||B| \end{aligned}$$

- If $R \neq I$ then $|AB| = 0 = 0|B|$

Matrix Inverse using Cofactors

Theorem

If A is an $n \times n$ matrix, then A is invertible if and only if $|A| \neq 0$.

Proof:

- implied by the first theorem of today: by (4) either R is I or it has a row of zeros.
- Note also that if A is invertible then $|AA^{-1}| = |A||A^{-1}| = |I|$. Hence $|A| \neq 0$ and

$$|A^{-1}| = \frac{1}{|A|}$$

- if $|A| \neq 0$ then A is invertible: we show this by construction:

Definition (Adjoint)

If A is an $n \times n$ matrix, the matrix of cofactors of A if the matrix whose (i,j) entry is C_{ij}, the (i,j) cofactor of A.
The adjoint or (adjugate) of A is the transpose of the matrix of cofactors, ie:

$$\mathrm{adj(A)} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- $$A\,\mathrm{adj(A)} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- entry $(1, 1)$ is $a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$, ie, cofactor by row 1
  entry $(1, 2)$ is $a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n}$, ie, entries of row 1 multiplied by cofactors of row 2

$$A\,\mathrm{adj(A)} = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix} = |A|I$$

- Since $|A| \neq 0$ we can divide:

$$A\left(\frac{1}{|A|}\mathrm{adj}(A)\right) = I \qquad A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$$

□

Outline

Matrix Inverse using Cofactors

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

What is A^{-1}?

- $|A| = 1(2-1) - 2(-1-4) + 3(-1-8) = -16 \neq 0 \implies$ invertible
- Matrix of cofactors

$$\begin{bmatrix} +M_{11} & -M_{12} & +M_{13} & -M_{14} & \cdots \\ -M_{21} & +M_{22} & -M_{23} & +M_{24} & \cdots \\ +M_{31} & -M_{32} & +M_{33} & -M_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -9 \\ 1 & -11 & 7 \\ -4 & 4 & 4 \end{bmatrix}$$

- $$A^{-1} = \frac{1}{|A|}\,\text{adj}(A) = -\frac{1}{16}\begin{bmatrix} 1 & 5 & -9 \\ 1 & -11 & 7 \\ -4 & 4 & 4 \end{bmatrix}^T = -\frac{1}{16}\begin{bmatrix} 1 & 1 & -4 \\ 5 & -11 & 4 \\ -9 & 7 & 4 \end{bmatrix}$$

Matrix Inverse using Cofactors

Example (cntd)

- Verify $AA^{-1} = I$:

$$-\frac{1}{16}\begin{bmatrix}1 & 2 & 3\\ -1 & 2 & 1\\ 4 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -4\\ 5 & -11 & 4\\ -9 & 7 & 4\end{bmatrix} = -\frac{1}{16}\begin{bmatrix}-16 & 0 & 0\\ 0 & -16 & 0\\ 0 & 0 & -16\end{bmatrix} = I$$

Cramer's rule

Theorem (Cramer's rule)

If A is $n \times n$, $|A| \neq 0$, and $\mathbf{b} \in \mathbb{R}^n$, then the solution $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ of the linear system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{|A_i|}{|A|},$$

where A_i is the matrix obtained from A by replacing the ith column with the vector $\mathbf{b}$.

Proof: Since $|A| \neq 0$, A^{-1} exists and we can solve for $\mathbf{x}$ by multiplying $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}. The $\mathbf{x} = A^{-1}\mathbf{b}$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\implies x_i = \frac{1}{|A|}(b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni})$, ie, cofactor expansion of column i of A with column i replaced by $\mathbf{b}$, ie, $|A_i|$ □

Matrix Inverse using Cofactors

Example

Use Cramer's rule to solve:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 7 \\ -x & + & 2y & + & z & = & -3 \\ 4x & + & y & + & z & = & 5 \end{array}$$

- In matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$$

- $|A| = -16 \neq 0$
- $$x = \frac{\begin{vmatrix} 7 & 2 & 3 \\ -3 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}}{|A|} = 1, \quad y = \frac{\begin{vmatrix} 1 & 7 & 3 \\ -1 & -3 & 1 \\ 4 & 5 & 1 \end{vmatrix}}{|A|} = -3, \quad z = \frac{\begin{vmatrix} 1 & 2 & 7 \\ -1 & 2 & -3 \\ 4 & 1 & 5 \end{vmatrix}}{|A|} = 4$$

Summary (1/2)

- There are three methods to solve $A\mathbf{x} = \mathbf{b}$ if A is $n \times n$ and $|A| \neq 0$:
  1. Gaussian elimination
  2. Matrix solution: find A^{-1}, then calculate $\mathbf{x} = A^{-1}\mathbf{b}$
  3. Cramer's rule
- There is one method to solve $A\mathbf{x} = \mathbf{b}$ if A is $m \times n$ and $m \neq n$ or if $|A| = 0$:
  1. Gaussian elimination
- There are two methods to find A^{-1}:
  1. using cofactors for the adjoint matrix
  2. by row reduction of $[A \mid I]$ to $[I \mid A^{-1}]$

Summary (2/2)

- If A is an $n \times n$ matrix, then the following statements are equivalent:
  1. A is invertible
  2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}$
  3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$
  4. the reduced row echelon form of A is I.
  5. $|A| \neq 0$
- Solving $A\mathbf{x} = \mathbf{b}$ in practice and at the computer:
  - via LU factorization (much quicker if one has to solve several systems with the same matrix A but different vectors b)
  - if A is symmetric positive definite matrix then Cholesky decomposition (twice as fast)
  - if A is large or sparse then iterative methods