DM559

Linear and Integer Programming

Lecture 6
Rank and Range
Vector Spaces

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Rank

2. Range

3. Vector Spaces

Survey

The problem from the survey:

$$\begin{cases} x + z = 1 \\ 3x + 4y + z = 1 \\ + 4y - 2z = -2 \end{cases}$$

$$\det(A) = 1 \begin{vmatrix} 4 & 1 \\ 4 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix} = 0$$

Hence we cannot solve by inverse nor by Cramer's rule. We proceed by Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 0 & 4 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x &= 1 - t \\ y &= \tfrac{1}{2}t - \tfrac{1}{2} \\ z &= t \end{aligned} \ , \forall t \in \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} t, \forall t \in \mathbb{R} \qquad \text{infinitely many solutions}$$

In Python:

```
import sympy as sy
import numpy as np
b=np.array([3,4,1])
a=np.array([1,0,1])
c=b-3*a
A=np.vstack([a,b,c])
M=sy.Matrix(A)
M.rref()
np.linalg.det(A) # 1.3322676295501906e-15
np.dot(np.linalg.inv(A),A) # array([[ 0., 0., -1.],[ 0., 1., 0.],[ 0., 0., 1.]])
np.linalg.solve(A,[1,1,-2]) # array([ 0., 0., 1.])
```

Hence Python for numerical reasons does not recognize the determinant to be null and solves the system returning only one particular solution.

Knoweldge of the theory of linear algebra is important to avoid mistakes!

Outline

1. Rank

2. Range

3. Vector Spaces

Rank

- Synthesis of what we have seen so far under the light of two new concepts: rank and range of a matrix
- We saw that:
  every matrix is row-equivalent to a matrix in reduced row echelon form.

Definition (Rank of Matrix)

The rank of a matrix A, $\text{rank}(A)$, is

- the number of non-zero rows, or equivalently
- the number of leading ones

in a row echelon matrix obtained from A by elementary row operations.

$\rightsquigarrow$ For an $m \times n$ matrix A,

$$\text{rank}\, A \leq \min\{m, n\},$$

where $\min\{m, n\}$ denotes the smaller of the two integers m and n.

Example

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 6 \end{bmatrix} \xrightarrow[R_3'=R_3-3R_1]{R_2'=R_2-2R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \end{bmatrix} \xrightarrow[R_3'=R_3-R_2]{R_2'=-R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightsquigarrow \operatorname{rank}(M) = 2$

Extension of the main theorem

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A *is invertible*
2. $A\mathbf{x} = \mathbf{b}$ *has a unique solution for any* $\mathbf{b} \in \mathbb{R}$
3. $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution,* $\mathbf{x} = \mathbf{0}$
4. *the reduced row echelon form of* A *is* I.
5. $|A| \neq 0$
6. *the rank of* A *is* n

Rank and Systems of Linear Equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 1 \\ 2x & + & 3y & & & = & 5 \\ 3x & + & 5y & + & z & = & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 4 \end{array}\right] \xrightarrow[]{\substack{R_2'=R_2-2R_1 \\ R_3'=R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow[]{\substack{R_2'=-R_2 \\ R_3'=R_3-R_2}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 1 \\ & & x & + & 2z & = & -3 \\ 0x & + & 0y & + & 0z & = & -2 \end{array}$$

It is inconsistent!

The last row is of the type $0 = a, a \neq 0$, that is, the augmenting matrix has a leading one in the last column
$\text{rank}(A) = 2 \neq \text{rank}(A \mid \mathbf{b}) = 3$

1. A system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of the augmented matrix is precisely the same as the rank of the matrix A.

2. If an $m \times n$ matrix A has rank m, the system of linear equations, $A\mathbf{x} = \mathbf{b}$, will be consistent for all $\mathbf{b} \in \mathbb{R}^n$

- Since A has rank m then there is a leading one in every row. Hence $[A \mid \mathbf{b}]$ cannot have a row $[0, 0, \ldots, 0, 1] \implies \operatorname{rank} A \nless \operatorname{rank}(A \mid \mathbf{b})$
- $[A \mid \mathbf{b}]$ has also m rows $\implies \operatorname{rank}(A) \ngtr \operatorname{rank}(A \mid \mathbf{b})$
- Hence, $\operatorname{rank}(A) = \operatorname{rank}(A \mid \mathbf{b})$

Example

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \operatorname{rank}(B) = 3$$

Any system $B\mathbf{x} = \mathbf{d}$ in 4 unknowns and 3 equalities with $\mathbf{d} \in \mathbb{R}^3$ is consistent.

Since $\operatorname{rank}(A)$ is smaller than the number of variables, then there is a non-leading variable. Hence infinitely many solutions!

Example

$$B = \begin{bmatrix} 1 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 3 & 0 & 4 & 0 & -28 \\ 0 & 0 & 1 & 2 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(B) = 3 < 5 = n$

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 3x_2 & & & + & 4x_4 & & & = & -28 \\ & & & & x_3 & + & 2x_4 & & & = & -14 \\ & & & & & & & & x_5 & = & 5 \end{array}$$

x_1, x_3, x_5 are leading variables; x_2, x_4 are non-leading variables (set them to $s, t \in \mathbb{R}$)

$$\begin{aligned} x_1 &= -28 - 3s - 4t \\ x_2 &= s \\ x_3 &= -14 - 2t \\ x_4 &= t \\ x_5 &= 5 \end{aligned} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -28 \\ 0 \\ -14 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} t$$

Summary

Let $A\mathbf{x} = \mathbf{b}$ be a general linear system in n variables and m equations:

- If $\text{rank}(A) = r < m$ and $\text{rank}(A \mid b) = r + 1$ then the system is inconsistent. (the row echelon form of the augmented matrix has a row $[0\ 0\ \ \dots\ \ 0\ 1]$)

- If $\text{rank}(A) = r = \text{rank}(A \mid \mathbf{b})$ then the system is consistent and there are $n - r$ free variables;
  if $r < n$ there are infinitely many solutions, if $r = n$ there are no free variables and the solution is unique

Let $A\mathbf{x} = \mathbf{0}$ be an homogeneous system in n variables and m equations, $\text{rank}(A) = r$ (always consistent):

- if $r < n$ there are infinitely many solutions, if $r = n$ there are no free variables and the solution is unique, $\mathbf{x} = \mathbf{0}$.

General solutions in vector notation

Example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -28 \\ 0 \\ -14 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} t, \quad \forall s, t \in \mathbb{R}$$

For $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} = \mathbf{p} + \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-r} \mathbf{v}_{n-r}, \quad \forall \alpha_i \in \mathbb{R}, i = 1, \ldots, n-r$$

Note:

- if $\alpha_i = 0, \forall i = 1, \ldots, n-r$ then $A\mathbf{p} = \mathbf{b}$, ie, $\mathbf{p}$ is a particular solution
- if $\alpha_1 = 1$ and $\alpha_i = 0, \forall i = 2, \ldots, n-r$ then

$$A(\mathbf{p} + \mathbf{v}_1) = \mathbf{b} \;\rightarrow\; A\mathbf{p} + A\mathbf{v}_1 = \mathbf{b} \;\xrightarrow{A\mathbf{p}=\mathbf{b}}\; A\mathbf{v}_1 = 0$$

Thus (recall that $\mathbf{x} = \mathbf{p} + \mathbf{z}$, $\mathbf{z} \in N(A)$):

- If A is an $m \times n$ matrix of rank r, the general solutions of $A\mathbf{x} = \mathbf{b}$ is the sum of:
  - a particular solution $\mathbf{p}$ of the system $A\mathbf{x} = \mathbf{b}$ and
  - a linear combination $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-r}\mathbf{v}_{n-r}$ of solutions $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{n-r}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$
- If A has rank n, then $A\mathbf{x} = \mathbf{0}$ only has the solution $\mathbf{x} = \mathbf{0}$ and so $A\mathbf{x} = \mathbf{b}$ has a unique solution: $\mathbf{p}$

Outline

1. Rank

2. Range

3. Vector Spaces

Range

Definition (Range of a matrix)

Let A be an $m \times n$ matrix, the range of A, denoted by $R(A)$, is the subset of $\mathbb{R}^m$ given by

$$R(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

That is, the range is the set of all vectors $\mathbf{y} \in \mathbb{R}^m$ of the form $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, or
all $\mathbf{y} \in \mathbb{R}^m$ for which the system $A\mathbf{x} = \mathbf{y}$ is consistent.

Recall, if $\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ is any vector in $\mathbb{R}^n$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \qquad i = 1, \ldots, n.$$

Then $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and

$$A\mathbf{x} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_n \mathbf{a}_n$$

that is, vector $A\mathbf{x}$ in $\mathbb{R}^n$ as a linear combination of the column vectors of A
Proof?

Hence $R(A)$ is the set of all linear combinations of the columns of A.
$\rightsquigarrow$ the range is also called the column space of A:

$$R(A) = \{\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_n \mathbf{a}_n \mid \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\}$$

Thus, $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is in the range of A, ie, a linear combination of the columns of A

Example

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

Then, for $\mathbf{x} = [\alpha_1, \alpha_2]^T$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \alpha_1 + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \alpha_2$$

so

$$R(A) = \left\{ \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} \middle| \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

Example

$$\begin{cases} \quad x + 2y = 0 \\ -x + 3y = -5 \\ 2x + y = 3 \end{cases}$$

$$\begin{cases} \quad x + 2y = 1 \\ -x + 3y = -5 \\ 2x + y = 2 \end{cases}$$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\mathbf{a}_1 - \mathbf{a}_2$$

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$. (Why?) Only way:

$$0 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0\mathbf{a}_1 + 0\mathbf{a}_2 = \mathbf{0}$$

Hence no way to express $[1, -5, 2]$ as linear expression of the two columns of A.

Outline

1. Rank

2. Range

3. Vector Spaces

Premise

- We move to a higher level of abstraction
- A vector space is a set with an addition and scalar multiplication that behave appropriately, that is, like $\mathbb{R}^n$
- Imagine a vector space as a class of a generic type (template) in object oriented programming, equipped with two operations.

Vector Spaces

Definition (Vector Space)

A (real) vector space V is a non-empty set equipped with an addition and a scalar multiplication operation such that for all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$:

1. $\mathbf{u} + \mathbf{v} \in V$ (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative law for addition)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associative law for addition)
4. there is a single member $\mathbf{0}$ of V, called the zero vector, such that for all $\mathbf{v} \in V, \mathbf{v} + \mathbf{0} = \mathbf{v}$
5. for every $\mathbf{v} \in V$ there is an element $\mathbf{w} \in V$, written $-\mathbf{v}$, called the negative of $\mathbf{v}$, such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
6. $\alpha\mathbf{v} \in V$ (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (distributive law)
8. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ (distributive law)
9. $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ (associative law for vector multiplication)
10. $1\mathbf{v} = \mathbf{v}$

Examples

- set $\mathbb{R}^n$
- but the set of objects for which the vector space defined is valid are more than the vectors in $\mathbb{R}^n$.
- set of all functions $F : \mathbb{R} \to \mathbb{R}$.
  We can define an addition $f + g$:
  $$(f + g)(x) = f(x) + g(x)$$
  and a scalar multiplication αf:
  $$(\alpha f)(x) = \alpha f(x)$$
- Example: $x + x^2$ and $2x$. They can represent the result of the two operations.
- What is $-f$? and the zero vector?

The axioms given are minimum number needed.
Other properties can be derived:
For example:

$$(-1)\mathbf{x} = -\mathbf{x}$$

$$\mathbf{0} = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = \mathbf{x} + (-1)\mathbf{x}$$

Adding $-\mathbf{x}$ on both sides:

$$-\mathbf{x} = -\mathbf{x} - \mathbf{0} = -\mathbf{x} + \mathbf{x} + (-1)\mathbf{x} = (-1)\mathbf{x}$$

which proves that $-\mathbf{x} = (-1)\mathbf{x}$.

Try the same with $-f$.

Examples

- $V = \{\mathbf{0}\}$
- the set of $m \times n$ all matrices
- the set of all infinite sequences of real numbers, $\mathbf{y} = \{y_1, y_2, \ldots, y_n, \ldots,\}, y_i \in \mathbb{R}$. ($\mathbf{y} = \{y_n\}$, $n \geq 1$)

  addition of $\mathbf{y} = \{y_1, y_2, \ldots, y_n, \ldots,\}$ and $\mathbf{z} = \{z_1, z_2, \ldots, z_n, \ldots,\}$ then:

  $$\mathbf{y} + \mathbf{z} = \{y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n, \ldots,\}$$

  multiplication by a scalar $\alpha \in \mathbb{R}$:

  $$\alpha\mathbf{y} = \{\alpha y_1, \alpha y_2, \ldots, \alpha y_n, \ldots,\}$$

- set of all vectors in $\mathbb{R}^3$ with the third entry equal to 0 (verify closure):

  $$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| \; x, y \in \mathbb{R} \right\}$$

Linear Combinations

Definition (Linear Combination)

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V, the vector

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k$$

is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
The scalars α_i are called coefficients.

- To find the coefficients that given a set of vertices express by linear combination a given vector, we solve a system of linear equations.
- If F is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ then the function $f : x \mapsto 2x^2 + 3x + 4$ can be expressed as a linear combination of:

  $$f = 2g + 3h + 4k$$

  where $g : x \mapsto x^2$, $h : x \mapsto x$, $k : x \mapsto 1$
- Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, is it possible to represent any point in the Cartesian plane?

Subspaces

Definition (Subspace)

A subspace W of a vector space V is a non-empty subset of V that is itself a vector space under the same operations of addition and scalar multiplication as V.

Theorem

Let V be a vector space. Then a non-empty subset W of V is a subspace if and only if both the following hold:

- *for all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$*
  (W is closed under addition)
- *for all $\mathbf{v} \in W$ and $\alpha \in \mathbb{R}$, $\alpha\mathbf{v} \in W$*
  (W is closed under scalar multiplication)

ie, all other axioms can be derived to hold true

Example

- The set of all vectors in $\mathbb{R}^3$ with the third entry equal to 0.
- The set $\{\mathbf{0}\}$ is not empty, it is a subspace since $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $\alpha\mathbf{0}=\mathbf{0}$ for any $\alpha \in \mathbb{R}$.

Example

In $\mathbb{R}^2$, the lines $y = 2x$ and $y = 2x + 1$ can be defined as the sets of vectors:

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = 2x, x \in \mathbb{R} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = 2x + 1, x \in \mathbb{R} \right\}$$

$$S = \{\mathbf{x} \mid \mathbf{x} = t\mathbf{v}, t \in \mathbb{R}\} \qquad U = \{\mathbf{x} \mid \mathbf{x} = \mathbf{p} + t\mathbf{v}, t \in \mathbb{R}\}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example (cntd)

1. The set S is non-empty, since $\mathbf{0} = 0\mathbf{v} \in S$.
2. closure under addition:

$$\mathbf{u} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S, \quad \mathbf{w} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S, \qquad \text{for some } s, t \in \mathbb{R}$$

$\mathbf{u} + \mathbf{w} = s\mathbf{v} + t\mathbf{v} = (s+t)\mathbf{v} \in S$ since $s + t \in \mathbb{R}$

3. closure under scalar multiplication:

$$\mathbf{u} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S \qquad \text{for some } s \in \mathbb{R}, \qquad \alpha \in \mathbb{R}$$

$\alpha\mathbf{u} = \alpha(s(\mathbf{v})) = (\alpha s)\mathbf{v} \in S$ since $\alpha s \in \mathbb{R}$

Note that:

- $\mathbf{u}, \mathbf{w}$ and $\alpha \in \mathbb{R}$ must be arbitrary

Example (cntd)

1. $\mathbf{0} \notin U$
2. U is not closed under addition:

$$\begin{bmatrix}0\\1\end{bmatrix} \in U, \begin{bmatrix}1\\3\end{bmatrix} \in U \qquad \text{but} \qquad \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix} \notin U$$

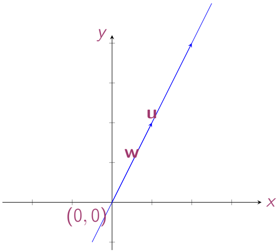
3. U is not closed under scalar multiplication

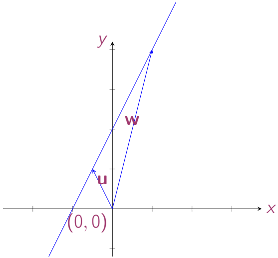
$$\begin{bmatrix}0\\1\end{bmatrix} \in U, 2 \in \mathbb{R} \qquad \text{but} \qquad 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \notin U$$

Note that:

- to prove just one of the above couterexamples suffices to show that U is not a subspace
- it is sufficient to make them fail for particular choices
- a good place to start is checking whether $\mathbf{0} \in S$. If not then S is not a subspace

Geometric interpretation:





⇝ The line $y = 2x + 1$ is an affine subset, a „translation" of a subspace

Theorem

A non-empty subset W of a vector space is a subspace if and only if for all $\mathbf{u}, \mathbf{v} \in W$ and all $\alpha, \beta \in \mathbb{R}$, we have $\alpha\mathbf{u} + \beta\mathbf{v} \in W$.
That is, W is closed under linear combination.

Summary

- Rank of a matrix and relation to number of solutions of a linear system
- General solutions of a linear system in vector notation
- Range, set of linear combinations of the columns of a matrix
- Vector spaces: properties
- Linear combination
- Subspaces: non-empty + closed under linear combination