DM559

Linear and Integer Programming

Lecture 7
Vector Spaces (cntd)
Linear Independence, Bases and Dimension

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Outline

Null space of a Matrix is a Subspace

Theorem

For any $m \times n$ matrix A, $N(A)$, ie, the solutions of $A\mathbf{x} = \mathbf{0}$, is a **subspace** *of* $\mathbb{R}^n$

Proof

1. $A\mathbf{0} = \mathbf{0} \quad \implies \quad \mathbf{0} \in N(A)$

2. Suppose $\mathbf{u}, \mathbf{v} \in N(A)$, then $\mathbf{u} + \mathbf{v} \in N(A)$:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

3. Suppose $\mathbf{u} \in N(A)$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{u} \in N(A)$:

$$A(\alpha\mathbf{u}) = A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha\mathbf{0} = \mathbf{0} \qquad \square$$

The set of solutions S to a general system $A\mathbf{x} = \mathbf{b}$ is not a **subspace** of $\mathbb{R}^n$ because $\mathbf{0} \notin S$

Affine subsets

Definition (Affine subset)

If W is a **subspace** of a **vector space** V and $\mathbf{x} \in V$, then the set $\mathbf{x} + W$ defined by

$$\mathbf{x} + W = \{\mathbf{x} + \mathbf{w} \mid \mathbf{w} \in W\}$$

is said to be an affine subset of V.

The set of solutions S to a general system $A\mathbf{x} = \mathbf{b}$ is an affine **subspace**, indeed recall that if $\mathbf{x}_0$ is any solution of the system

$$S = \{\mathbf{x}_0 + \mathbf{z} \mid \mathbf{z} \in N(A)\}$$

Range of a Matrix is a Subspace

Theorem

For any $m \times n$ matrix A, $R(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$ is a **subspace** *of $\mathbb{R}^m$*

Proof

1. $A\mathbf{0} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{0} \in R(A)$
2. Suppose $\mathbf{u}, \mathbf{v} \in R(A)$, then $\mathbf{u} + \mathbf{v} \in R(A)$:
   ...
3. Suppose $\mathbf{u} \in R(A)$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{u} \in R(A)$:
   ...

Linear Span

- If $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_k\mathbf{v}_k$, then $\mathbf{v} + \mathbf{w}$ and $s\mathbf{v}, s \in \mathbb{R}$ are also linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
- The set of all linear combinations of a given set of vectors of a **vector space** V forms a **subspace**:

Definition (Linear span)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. The linear span of $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, denoted by $\mathrm{Lin}(X)$, that is:

$$\mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}$$

Theorem

*If $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors of a vectors space V, then $\mathrm{Lin}(X)$ is a **subspace** of V and is also called the **subspace** spanned by X.*
*It is the smallest **subspace** containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.*

Example

- $\text{Lin}(\{\mathbf{v}\}) = \{\alpha\mathbf{v} \mid \alpha \in \mathbb{R}\}$ defines a line in $\mathbb{R}^n$.
- Recall that a plane in $\mathbb{R}^3$ has two equivalent representations:

$$ax + by + cz = d \qquad \text{and} \qquad \mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

  where $\mathbf{v}$ and $\mathbf{w}$ are non parallel.

  - If $d = 0$ and $\mathbf{p} = \mathbf{0}$, then

$$\{\mathbf{x} \mid \mathbf{x} = s\mathbf{v} + t\mathbf{w}, s, t, \in \mathbb{R}\} = \text{Lin}(\{\mathbf{v}, \mathbf{w}\})$$

    and hence a **subspace** of $\mathbb{R}^n$.

  - If $d \neq 0$, then the plane is not a **subspace**. It is an affine subset, a translation of a **subspace**.

    (recall that one can also show directly that a subset is a **subspace** or not)

Spanning Sets of a Matrix

Definition (Column space)

If A is an $m \times n$ matrix, and if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ denote the columns of A, then the column space or range of A is

$$CS(A) = R(A) = \mathrm{Lin}(\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\})$$

and is a **subspace** of $\mathbb{R}^m$.

Definition (Row space)

If A is an $m \times n$ matrix, and if $\overrightarrow{\mathbf{a}}_1, \overrightarrow{\mathbf{a}}_2, \ldots, \overrightarrow{\mathbf{a}}_k$ denote the rows of A, then the row space of A is

$$RS(A) = \mathrm{Lin}(\{\overrightarrow{\mathbf{a}}_1, \overrightarrow{\mathbf{a}}_2, \ldots, \overrightarrow{\mathbf{a}}_k\})$$

and is a **subspace** of $\mathbb{R}^n$.

- If A is an $m \times n$ matrix, then for any $\mathbf{r} \in RS(A)$ and any $\mathbf{x} \in N(A)$, $\langle \mathbf{r}, \mathbf{x} \rangle = 0$; that is, $\mathbf{r}$ and $\mathbf{x}$ are orthogonal. (hint: look at $A\mathbf{x} = \mathbf{0}$)

Summary

We have seen:

- Definition of **vector space** and **subspace**
- Proofs that a given set is a vector space
- Proofs that a given subset of a vector space is a **subspace** or not
- Definition of linear span of set of vectors
- Definition of row and column spaces of a matrix
  $CS(A) = R(A)$ and $RS(A) \perp N(A)$

Outline

Linear Independence

Definition (Linear Independence)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent (or form a linearly independent set) if and only if the vector equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

has the unique solution

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Definition (Linear Dependence)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent (or form a linearly dependent set) if and only if there are real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

Example

In $\mathbb{R}^2$, the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are linearly independent. Indeed:

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} \alpha + \beta = 0 \\ 2\alpha - \beta = 0 \end{cases}$$

The homogeneous linear system has only the trivial solution, $\alpha = 0, \beta = 0$, so linear independence.

Example

In $\mathbb{R}^3$, the following vectors are linearly dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

Indeed: $2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$

Theorem

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$ is linearly dependent if and only if at least one vector $\mathbf{v}_i$ is a linear combination of the other vectors.

Proof

$\Longrightarrow$

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are linearly dependent then

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

has a solution with some $\alpha_i \neq 0$, then:

$$\mathbf{v}_i = -\frac{\alpha_1}{\alpha_i}\mathbf{v}_1 - \frac{\alpha_2}{\alpha_i}\mathbf{v}_2 - \cdots - \frac{\alpha_{i-1}}{\alpha_i}\mathbf{v}_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}\mathbf{v}_{i+1} + \cdots - \frac{\alpha_k}{\alpha_i}\mathbf{v}_k$$

which is a linear combination of the other vectors

$\Longleftarrow$

If $\mathbf{v}_i$ is a lin combination of the other vectors, eg,

$$\mathbf{v}_i = \beta_1 \mathbf{v}_1 + \cdots + \beta_{i-1}\mathbf{v}_{i-1} + \beta_{i+1}\mathbf{v}_{i+1} + \cdots + \beta_k \mathbf{v}_k$$

then

$$\beta_1 \mathbf{v}_1 + \cdots + \beta_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + \beta_{i+1}\mathbf{v}_{i+1} + \cdots + \beta_k \mathbf{v}_k = \mathbf{0}$$ □

Corollary

Two vectors are linearly dependent if and only if at least one vector is a scalar multiple of the other.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

are linearly independent

Theorem

In a **vector space** V*, a non-empty set of vectors that contains the zero vector is linearly dependent.*

Proof:

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset V$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{0}\}$$

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_k + a\mathbf{0} = \mathbf{0}, \qquad a \neq 0$$

Uniqueness of linear combinations

Theorem

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent vectors in V and if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_k\mathbf{v}_k$$

then

$$a_1 = b_1, \quad a_2 = b_2, \quad \ldots \quad a_k = b_k.$$

- If a vector $\mathbf{x}$ can be expressed as a linear combination of linearly independent vectors, then this can be done in only one way

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

Testing for Linear Independence in $\mathbb{R}^n$

For k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$$

is equivalent to

$$A\mathbf{x}$$

where A is the $n \times k$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and $\mathbf{x} = [\alpha_1, \alpha_2, \ldots, \alpha_k]^T$:

Theorem

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly dependent if and only if the linear system $A\mathbf{x} = \mathbf{0}$, where A is the matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_k]$, has a solution other than $\mathbf{x} = \mathbf{0}$.
Equivalently, the vectors are linearly independent precisely when the only solution to the system is $\mathbf{x} = \mathbf{0}$.

If vectors are linearly dependent, then any solution $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} = [\alpha_1, \alpha_2 \ldots, \alpha_k]^T$ of $A\mathbf{x} = \mathbf{0}$ gives a non-trivial linear combination $A\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k = \mathbf{0}$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

are linearly dependent.
We solve $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -5 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

The general solution is

$$\mathbf{v} = \begin{bmatrix} t \\ -3t \\ t \end{bmatrix}$$

and $A\mathbf{x} = t\mathbf{v}_1 - 3t\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$

Hence, for $t = 1$ we have: $\quad 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Recall that $A\mathbf{x} = \mathbf{0}$ has precisely one solution $\mathbf{x} = \mathbf{0}$ iff the $n \times k$ matrix is row equiv. to a row echelon matrix with k leading ones, ie, iff $\text{rank}(A) = k$

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent iff the $n \times k$ matrix $A = [\mathbf{v}_1 \; \mathbf{v}_2 \; \ldots \; \mathbf{v}_k]$ has rank k.

Theorem

The maximum size of a linearly independent set of vectors in $\mathbb{R}^n$ is n.

- $\text{rank}(A) \leq \min\{n, k\}$, hence $\text{rank}(A) \leq n \Rightarrow$ when lin. indep. $k \leq n$.
- we exhibit an example that has exactly n independent vectors in $\mathbb{R}^n$ (there are infinite examples):

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This is known as the standard basis of $\mathbb{R}^n$.

Example

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 1 \end{bmatrix} \right\} \quad \text{lin. dep. since } 5 > n = 4$$

$$L_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix} \right\} \quad \text{lin. indep.}$$

$$L_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\} \quad \text{lin. dep. since } \operatorname{rank}(A) = 2$$

$$L_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{lin. dep. since } L_3 \subseteq L_4$$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set of vectors in a **vector space** *V and if $\mathbf{w} \in V$ is not in the linear span of S, ie, $\mathbf{w} \notin \mathrm{Lin}(s)$, then the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is linearly independent.*

Proof:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k + b\mathbf{w} = \mathbf{0}$$

If $b \neq 0$, then we solve for $\mathbf{w}$ and find that it is a linear combination: contradiction, $\mathbf{w} \notin \mathrm{Lin}(S)$.

Hence $b = 0$ and $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k = \mathbf{0}$ implies by hypothesis that all α_i are zero. $\square$

Linear Independence and Span in $\mathbb{R}^n$

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.
What are the conditions for S to span $\mathbb{R}^n$ and be linearly independent?

Let A be the $n \times k$ matrix whose columns are the vectors from S.

- S spans $\mathbb{R}^n$ if for any $v \in \mathbb{R}^n$ the linear system $A\mathbf{x} = \mathbf{v}$ is consistent. This happens when $\text{rank}(A) = n$, hence $k \geq n$
- S is linearly independent iff the linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution. This happens when $\text{rank}(A) = k$, Hence $k \leq n$

Hence, to span $\mathbb{R}^n$ and to be linearly independent, the set S must have exactly n vectors and the square matrix A must have $\det(A) \neq 0$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 5 & 1 \end{vmatrix} = 30 \neq 0$$

Outline

1. Vector Spaces (cntd)

2. Linear independence

3. Bases

4. Dimension

Bases

Definition (Basis)

Let V be a **vector space**. Then the subset $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of V is said to be a basis for V if:

1. B is a linearly independent set of vectors, and
2. B spans V; that is, $V = \mathrm{Lin}(B)$

Theorem

*If V is a **vector space**, then a smallest spanning set is a basis of V.*

Theorem

$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V if and only if any $\mathbf{v} \in V$ is a unique linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Example

$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.
the vectors are linearly independent and for any $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$, ie,

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example

The set below is a basis of $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

- any vector $\mathbf{x} \in \mathbb{R}^2$ can be written as a linear combination of vectors in S.
- any vector $\mathbf{b}$ is a linear combination of the two vectors in S
  $\rightsquigarrow A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$.
- S spans $\mathbb{R}^2$ and is linearly independent

Example

Find a basis of the **subspace** of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y - 3z = 0 \right\}.$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x + 3z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = x\mathbf{v} + z\mathbf{w}, \quad \forall x, z \in \mathbb{R}$$

The set $\{\mathbf{v}, \mathbf{w}\}$ spans W. The set is also independent:

$$\alpha\mathbf{v} + \beta\mathbf{w} = \mathbf{0} \implies \alpha = 0, \beta = 0$$

Extension of the main theorem

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. *A is invertible*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}$*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$*
4. *the reduced row echelon form of A is I.*
5. *$|A| \neq 0$*
6. *The rank of A is n*
7. *The column vectors of A are a basis of $\mathbb{R}^n$*
8. *The rows of A (written as vectors) are a basis of $\mathbb{R}^n$*

(The last statement derives from $|A^T| = |A|$.)
Hence, simply calculating the determinant can inform on all the above facts.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

This set is linearly dependent since $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$
so $\mathbf{v}_3 \in \text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\})$ and $\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.
The linear span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^3$ is a plane:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2 = s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The vector $\mathbf{x}$ belongs to the **subspace** iff it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, that is, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}$ are linearly dependent or:

$$|A| = \begin{vmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 3 & 5 & z \end{vmatrix} = 0 \qquad \Longrightarrow \qquad |A| = 7x + y - 3z = 0$$

Coordinates

Definition (Coordinates)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V, then any vector $\mathbf{v} \in V$ can be expressed uniquely as $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ then the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coordinates of $\mathbf{v}$ with respect to the basis S. We use the notation

$$[\mathbf{v}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_S$$

to denote the coordinate vector of $\mathbf{v}$ in the basis S.

Example

Consider the two basis of $\mathbb{R}^2$:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}_B \qquad [\mathbf{v}]_S = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_S$$

In the standard basis the coordinates of $\mathbf{v}$ are precisely the components of the vector $\mathbf{v}$.
In the basis S, they are such that

$$\mathbf{v} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Outline

1. Vector Spaces (cntd)

2. Linear independence

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4. Dimension

Dimension

Theorem

Let V be a **vector space** *with a basis*

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

of n vectors. Then any set of $n+1$ vectors is linearly dependent.

Proof:

- Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n+1}\}$ be any set of $n+1$ vectors in V.
- Since B is a basis, then

$$\mathbf{w}_i = a_{1i}\mathbf{v}_1 + a_{2i}\mathbf{v}_2 + \ldots + a_{ni}\mathbf{v}_n$$

- linear combination of vectors in S:

$$b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \cdots + b_{n+1}\mathbf{w}_{n+1} = \mathbf{0}$$

Substituting:

$$b_1(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \ldots + a_{n1}\mathbf{v}_n) + b_2(a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \ldots + a_{n2}\mathbf{v}_n) + \cdots$$
$$+ b_{n+1}(a_{1,n+1}\mathbf{v}_1 + a_{2,n+1}\mathbf{v}_2 + \ldots + a_{n,n+1}\mathbf{v}_n) = \mathbf{0}$$

$$b_1(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \ldots + a_{n1}\mathbf{v}_n) + b_2(a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \ldots + a_{n2}\mathbf{v}_n) + \cdots$$
$$+ b_{n+1}(a_{1,n+1}\mathbf{v}_1 + a_{2,n+1}\mathbf{v}_2 + \ldots + a_{n,n+1}\mathbf{v}_n) = \mathbf{0}$$

collecting the terms that multiply the vectors:

$$(b_1a_{11}+b_2a_{12}+\cdots+b_{n+1}a_{1,n+1})\mathbf{v}_1+(b_1a_{2,1}+b_2a_{2,2}+\cdots+b_{n+1}a_{2,n+1})\mathbf{v}_2+\cdots$$
$$+ (b_1a_{n,1} + b_2a_{n,2} + \cdots + b_{n+1}a_{n,n+1})\mathbf{v}_n = \mathbf{0}$$

this gives us the system

$$\begin{cases} b_1a_{11} + b_2a_{12} + \cdots + b_{n+1}a_{1,n+1} = 0 \\ b_1a_{2,1} + b_2a_{2,2} + \cdots + b_{n+1}a_{2,n+1} = 0 \\ \vdots \\ b_1a_{n,1} + b_2a_{n,2} + \cdots + b_{n+1}a_{n,n+1} = 0 \end{cases}$$

Homogeneous system of $n+1$ variables $(b_1, \ldots, b_{n+1})$ in n equations. Hence at least one free variable. Hence

$$b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \cdots + b_{n+1}\mathbf{w}_{n+1} = \mathbf{0}$$

has non trivial solutions and the set S is linearly dependent.

It follows that:

Theorem

Let a **vector space** V *have a finite basis consisting of* r *vectors. Then any basis of* V *consists of exactly* r *vectors.*

Definition (Dimension)

The number of k vectors in a finite basis of a **vector space** V is the dimension of V and is denoted by $\dim(V)$.
The **vector space** $V = \{\mathbf{0}\}$ is defined to have dimension 0.

- a plane in $\mathbb{R}^2$ is a two-dimensional **subspace**
- a line in $\mathbb{R}^n$ is a one-dimensional **subspace**
- a hyperplane in $\mathbb{R}^n$ is an $(n-1)$-dimensional **subspace** of $\mathbb{R}^n$
- the **vector space** F of real functions is an infinite-dimensional **vector space**
- the **vector space** of real-valued sequences is an infinite-dimensional **vector space**.

Dimension and bases of Subspaces

Example

The plane W in $\mathbb{R}^3$

$$W = \{\mathbf{x} \mid x + y - 3z = 0\}$$

has a basis consisting of the vectors $\mathbf{v}_1 = [1, 2, 1]^T$ and $\mathbf{v}_2 = [3, 0, 1]^T$.

Let $\mathbf{v}_3$ be any vector $\notin W$, eg, $\mathbf{v}_3 = [1, 0, 0]^T$. Then the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

Basis and Dimension in $\mathbb{R}^n$

Recall: Three **subspace**s associated with an $m \times n$ matrix A:

$RS(A)$ row space: linear span of the rows of A
subspace of $\mathbb{R}^n$

$N(A)$ null space: set of all solutions of $A\mathbf{x} = \mathbf{0}$
subspace of $\mathbb{R}^n$

$R(A) = CS(A)$ range or column space: linear span of column vectors;
subspace of $\mathbb{R}^m$

How do we find a basis for these subspaces?

Example

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 4 \\ -1 & 3 & 9 & 1 & 9 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix}$$

$$RS(A) = \text{Lin}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 9 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}\right) \qquad \text{subspace in } \mathbb{R}^5$$

$$N(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} \qquad \text{subspace in } \mathbb{R}^5$$

$$R(A) = CS(A) = \text{Lin}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 9 \\ 1 \end{bmatrix} \right\}\right) \qquad \text{subspace in } \mathbb{R}^4$$

Example (cntd)

$$A \to \cdots \to \begin{bmatrix} 1 & 0 & -3 & 0 & -3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$RS(A) = RS(R)$ because row operations are linear combinations of the vectors. Hence a basis for $RS(A)$ is given by the non-zero rows:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

it is a three-dimensional **subspace** of $\mathbb{R}^5$

Example (cntd)

$$A \to \cdots \to \begin{bmatrix} 1 & 0 & -3 & 0 & -3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Basis for $N(A)$. We write the general solution for $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3s + 3t \\ -2s - t \\ s \\ -3t \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2, \qquad s, t \in \mathbb{R}$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis since also linearly independent
It is a two-dimensional **subspace** of $\mathbb{R}^5$

Example (cntd)

$$A \to \cdots \to \begin{bmatrix} 1 & 0 & -3 & 0 & -3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$R(A) = CS(A)$. operations on rows, but vectors are the columns. However the columns that have a leading one are columns that are linearly independent, because the RREF of the corresponding columns of A would have only one leading one is in every column.
The basis is $\{a_1, a_2, a_4\}$, ie, the three columns of the starting matrix

Any other vector added would be dependent

It is a three-dimensional **subspace** of $\mathbb{R}^4$

Basis of a Linear Space

If we are given k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, how can we find a basis for $\mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\})$?

We can:

- create an $k \times n$ matrix (vectors as rows) and find a basis for the row space or
- create an $n \times k$ matrix (vectors as columns) and find a basis for the column space.

For both cases we put the matrix in reduced row echelon form.

Definition (Rank and nullity)

The rank of a matrix A is

$$\text{rank}(A) = \dim(R(A))$$

The nullity of a matrix A is

$$\text{nullity}(A) = \dim(N(A))$$

Although **subspaces** of possibly different Euclidean spaces:

Theorem

If A is an $m \times n$ matrix, then

$$\dim(RS(A)) = \dim(CS(A)) = \text{rank}(A)$$

Theorem (Rank-nullity theorem)

For an $m \times n$ matrix A

$$\text{rank}(A) + \text{nullity}(A) = n \qquad (\dim(R(A)) + \dim(N(A)) = n)$$

Summary

- Linear dependence and independence
- Determine linear dependency of a set of vertices, ie, find non-trivial lin. combination that equal zero
- Basis
- Find a basis for a linear space
- Find a basis for the null space, range and row space of a matrix (from its reduced echelon form)
- Dimension (finite, infinite)
- Rank-nullity theorem