DM559

Linear and Integer Programming

Lecture 8
Linear Transformations

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Outline

Resume

- Linear dependence and independence
- Determine linear dependency of a set of vertices, ie, find non-trivial lin. combination that equal zero
- Basis
- Find a basis for a linear space
- Find a basis for the null space, range and row space of a matrix (from its reduced echelon form)
- Dimension (finite, infinite)
- Rank-nullity theorem

Outline

Linear Transformations

Definition (Linear Transformation)

Let V and W be two vector spaces. A function $T : V \to W$ is linear if for all $\mathbf{u}, \mathbf{v} \in V$ and all $\alpha \in \mathbb{R}$:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$

A linear transformation is a linear function between two vector spaces

- If $V = W$ also known as linear operator
- Equivalent condition: $T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$
- for all $\mathbf{0} \in V$, $T(\mathbf{0}) = \mathbf{0}$

Example (Linear Transformations)

- vector space $V = \mathbb{R}$, $F_1(x) = px$ for any $p \in \mathbb{R}$

$$\begin{aligned} \forall x, y \in \mathbb{R}, \alpha, \beta \in \mathbb{R} : F_1(\alpha x + \beta y) &= p(\alpha x + \beta y) = \alpha(px) + \beta(px) \\ &= \alpha F_1(x) + \beta F_1(y) \end{aligned}$$

- vector space $V = \mathbb{R}$, $F_1(x) = px + q$ for any $p, q \in \mathbb{R}$ or $F_3(x) = x^2$ are not linear transformations

$$T(x+y) \neq T(x) + T(y) \forall x, y \in \mathbb{R}$$

- vector spaces $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $m \times n$ matrix A, $T(x) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}) \\ T(\alpha\mathbf{u}) &= A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha T(\mathbf{u}) \end{aligned}$$

Example (Linear Transformations)

- vector spaces $V = \mathbb{R}^n$, $W : f : \mathbb{R} \to \mathbb{R}$. $T : \mathbb{R}^n \to W$:

$$T(\mathbf{u}) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) = p_{u_1, u_2, \ldots, u_n} = p_{\mathbf{u}}$$

$$p_{u_1, u_2, \ldots, u_n} = u_1 x^1 + u_2 x^2 + u_3 x^3 + \cdots + u_n x^n$$

$$\begin{aligned} p_{\mathbf{u}+\mathbf{v}}(x) &= \cdots = (p_{\mathbf{u}} + p_{\mathbf{v}})(x) \\ p_{\alpha \mathbf{u}}(\mathbf{x}) &= \cdots = \alpha p_u(x) \end{aligned}$$

Linear Transformations and Matrices

- any $m \times n$ matrix A defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow T_A$
- for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is a matrix A such that $T(\mathbf{v}) = A\mathbf{v} \rightsquigarrow A_T$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ denote the standard basis of $\mathbb{R}^n$ and let A be the matrix whose columns are the vectors $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$: that is,

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix}$$

Then, for every $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$.

Proof: write any vector $\mathbf{x} \in \mathbb{R}^n$ as lin. comb. of standard basis and then make the image of it.

Example

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ x - y \\ x + 2y - 3z \end{bmatrix}$$

- The image of $\mathbf{u} = [1, 2, 3]^T$ can be found by substitution: $T(\mathbf{u}) = [6, -1, -4]^T$.

- to find A_T:

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_n)] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}$$

$T(\mathbf{u}) = A\mathbf{u} = [6, -1, -4]^T$.

Linear Transformation in $\mathbb{R}^2$

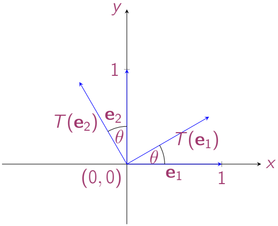
- We can visualize them!
- Reflection in the x axis:

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix} \qquad A_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Stretching the plane away from the origin

$$T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Rotation anticlockwise by an angle θ



we search the images of the standard basis vector $\mathbf{e}_1, \mathbf{e}_2$

$$T(\mathbf{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix}, \quad T(\mathbf{e}_1) = \begin{bmatrix} d \\ b \end{bmatrix}$$

they will be orthogonal and with length 1.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For $\pi/4$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Identity and Zero Linear Transformations

- For $T : V \to V$ the linear transformation such that $T(\mathbf{v}) = \mathbf{v}$ is called the identity.
- if $V = \mathbb{R}^n$, the matrix $A_T = I$ (of size $n \times n$)
- For $T : V \to W$ the linear transformation such that $T(\mathbf{v}) = \mathbf{0}$ is called the zero transformation.
- If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, the matrix A_T is an $m \times n$ matrix of zeros.

Composition of Linear Transformations

- Let $T : V \to W$ and $S : W \to U$ be linear transformations.
  The composition of ST is again a linear transformation given by:

  $$ST(\mathbf{v}) = S(T(\mathbf{v})) = S(\mathbf{w}) = \mathbf{u}$$

  where $\mathbf{w} = T(\mathbf{v})$

- ST means do T and then do S: $V \xrightarrow{T} W \xrightarrow{S} U$

- if $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ in terms of matrices:

  $$ST(\mathbf{v}) = S(T(\mathbf{v})) = S(A_T\mathbf{v}) = A_S A_T \mathbf{v}$$

  note that composition is not commutative

Combinations of Linear Transformations

- If $S, T : V \to W$ are linear transformations between the same vector spaces, then $S + T$ and αS, $\alpha \in \mathbb{R}$ are linear transformations.

- hence also $\alpha S + \beta T$, $\alpha, \beta \in \mathbb{R}$ is

Inverse Linear Transformations

- If V and W are finite-dimensional vector spaces of the same dimension, then the inverse of a lin. transf. $T : V \to W$ is the lin. transf such that

  $$T^{-1}(T(v)) = \mathbf{v}$$

- In $\mathbb{R}^n$ if T^{-1} exists, then its matrix satisfies:

  $$T^{-1}(T(v)) = A_{T^{-1}} A_T \mathbf{v} = I\mathbf{v}$$

  that is, T^{-1} exists iff $(A_T)^{-1}$ exists and $A_{T^{-1}} = (A_T)^{-1}$
  (recall that if $BA = I$ then $B = A^{-1}$)

- In $\mathbb{R}^2$ for rotations:

  $$A_{T^{-1}} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Example

Is there an inverse to $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ x - y \\ x + 2y - 3z \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}$$

Since $\det(A) = 9$ then the matrix is invertible, and T^{-1} is given by the matrix:

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 5 & 1 \\ 3 & -4 & 1 \\ 3 & -1 & -2 \end{bmatrix} \qquad T^{-1}\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3}u + \frac{5}{9}v + \frac{1}{9}w \\ \frac{1}{3}u - \frac{4}{9}v + \frac{1}{9}w \\ \frac{1}{3}u + \frac{1}{9}v - \frac{2}{9}w \end{bmatrix}$$

Linear Transformations from V to W

Theorem

Let V be a finite-dimensional vector space and let T be a linear transformation from V to a vector space W.
Then T is completely determined by what it does to a basis of V.

Proof

(unique representation in V implies unique representation in T)

- If both V and W are finite dimensional vector spaces, then we can find a matrix that represents the linear transformation:

- suppose V has $\dim(V) = n$ and basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
  and W has $\dim(W) = m$ and basis $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$;

- coordinates of $\mathbf{v} \in V$ are $[\mathbf{v}]_B$
  coordinates of $T(\mathbf{v}) \in W$ are $[T(\mathbf{v})]_S$

- we search for a matrix A such that:

$$[T(\mathbf{v})]_S = A[\mathbf{v}]_B$$

- we find it by:

$$\begin{aligned}[T(\mathbf{v})]_S &= a_1[T(\mathbf{v}_1)]_S + a_2[T(\mathbf{v}_2)]_S + \cdots + a_n[T(\mathbf{v}_n)]_S \\ &= [[T(\mathbf{v}_1)]_S \; [T(\mathbf{v}_2)]_S \; \cdots \; [T(\mathbf{v}_n)]_S] \, [\mathbf{v}]_B\end{aligned}$$

  where $[\mathbf{v}]_B = [a_1, a_2, \ldots, a_n]^T$

Range and Null Space

Definition (Range and null space)

$T : V \to W$. The range $R(T)$ of T is:

$$R(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$$

and the null space (or kernel) $N(T)$ of T is

$$N(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

- the range is a subspace of W and the null space of V.
- Matrix case, $T : \mathbb{R}^n \to \mathbb{R}^m$
  $R(T) = R(A)$ $N(T) = N(A)$
- Rank-nullity theorem:
  $\text{rank}(T) = \dim(R(T))$
  $\text{nullity}(T) = \dim(N(T))$

  $$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Example

Construct a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$N(T) = \left\{ t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \;:\; t \in \mathbb{R} \right\}, \qquad R(T) = xy\text{-plane}.$$

Outline

1. Linear Transformations

2. Coordinate Change

Coordinates

Recall:

Definition (Coordinates)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis of a vector space V, then

- any vector $\mathbf{v} \in V$ can be expressed uniquely as $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$
- and the real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are the coordinates of $\mathbf{v}$ wrt the basis S.

To denote the coordinate vector of $\mathbf{v}$ in the basis S we use the notation

$$[\mathbf{v}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_S$$

- In the standard basis the coordinates of $\mathbf{v}$ are precisely the components of the vector $\mathbf{v}$: $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$
- How to find coordinates of a vector $\mathbf{v}$ wrt another basis?

Transition from Standard to Basis B

Definition (Transition Matrix)

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$. The coordinates of a vector $\mathbf{x}$ wrt B, $\mathbf{a} = [a_, a_2, \ldots, a_n]^T = [\mathbf{x}]_B$, are found by solving the linear system:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = \mathbf{x} \qquad \text{that is} \qquad \mathbf{x} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \mathbf{v}_n]\mathbf{a}$$

We call P the matrix whose columns are the basis vectors:

$$P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \mathbf{v}_n]$$

Then for any vector $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{x} = P[\mathbf{x}]_B$$ transition matrix from B coords to standard coords

moreover P is invertible (columns are a basis):

$$[\mathbf{x}]_B = P^{-1}\mathbf{x}$$ transition matrix from standard coords to B coords

Example

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \qquad [\mathbf{v}]_B = \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$\det(P) = 4 \neq 0$ so B is a basis of $\mathbb{R}^3$ standard coordinates of $\mathbf{v}$:

$$\mathbf{v} = 4 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - 5 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \\ -5 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}_B = \begin{bmatrix} -9 \\ -3 \\ -5 \end{bmatrix}$$

Example (cntd)

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}, \qquad [\mathbf{x}] = \begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix}$$

B coordinates of vector $\mathbf{x}$:

$$\begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

either we solve $P\mathbf{a} = \mathbf{x}$ in $\mathbf{a}$ by Gaussian elimination or we find the inverse P^{-1}:

$$[\mathbf{x}]_B = P^{-1}\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}_B \qquad \text{check the calculation}$$

What are the B coordinates of the basis vector? ($[1, 0, 0], [0, 1, 0], [0, 0, 1]$)

Change of Basis

Since $T(\mathbf{x}) = P\mathbf{x}$ then $T(\mathbf{e}_i) = \mathbf{v}_i$, ie, T maps standard basis vector to new basis vectors

Example

Rotate basis in $\mathbb{R}^2$ by $\pi/4$ anticlockwise, find coordinates of a vector wrt the new basis.

$$A_T = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since the matrix A_T rotates $\{\mathbf{e}_1, \mathbf{e}_2\}$, then $A_T = P$ and its columns tell us the coordinates of the new basis and $\mathbf{v} = P[\mathbf{v}]_B$ and $[\mathbf{v}]_B = P^{-1}\mathbf{v}$. The inverse is a rotation clockwise:

$$P^{-1} = \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Example (cntd)

Find the new coordinates of a vector $\mathbf{x} = [1, 1]^T$

$$[\mathbf{x}]_B = P^{-1}\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Change of basis from B to B'

Given a basis B of $\mathbb{R}^n$ with transition matrix P_B,
and another basis B' with transition matrix $P_{B'}$,
how do we change from coords in the basis B to coords in the basis B'?

coordinates in B $\xrightarrow{\mathbf{v}=P_B[\mathbf{v}]_B}$ standard coordinates $\xrightarrow{[\mathbf{v}]_{B'}=P_{B'}^{-1}\mathbf{v}}$ coordinates in B'

$$[\mathbf{v}]_{B'} = P_{B'}^{-1} P_B [\mathbf{v}]_B$$

$$M = P_{B'}^{-1} P_B = P_{B'}^{-1} [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n] \overset{ex7sh3}{=} [P_{B'}^{-1}\mathbf{v}_1 \ P_{B'}^{-1}\mathbf{v}_2 \ \ldots \ P_{B'}^{-1}\mathbf{v}_n]$$

Theorem

If B and B' are two bases of $\mathbb{R}^n$, with

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

then the transition matrix from B coordinates to B' coordinates is given by

$$M = \left[[\mathbf{v}_1]_{B'} \ \ [\mathbf{v}_2]_{B'} \ \ \cdots \ \ [\mathbf{v}_n]_{B'} \right]$$

(the columns of M are the B' coordinates of the basis B)

Example

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \qquad S = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

are basis of $\mathbb{R}^2$, indeed the corresponding transition matrices from standard basis:

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

have $\det(P) = 3$, $\det(Q) = 1$. Hence, lin. indep. vectors.
We are given

$$[\mathbf{x}]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}_B$$

find its coordinates in S.

Example (cntd)

1. find first the standard coordinates of $\mathbf{x}$

$$\mathbf{x} = 4\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$$

   and then find S coordinates:

$$[\mathbf{x}]_S = Q^{-1}\mathbf{x} = \begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix} = \begin{bmatrix}-25\\16\end{bmatrix}_S$$

2. use transition matrix M from B to S coordinates:
   $\mathbf{v} = P[\mathbf{v}]_B$ and $\mathbf{v} = Q[\mathbf{v}]_S$ $\rightsquigarrow$ $[\mathbf{v}]_S = Q^{-1}P[\mathbf{v}]_B$:

$$M = Q^{-1}P = \begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix} = \begin{bmatrix}-8 & -7\\5 & 4\end{bmatrix}$$

$$[\mathbf{x}]_S = \begin{bmatrix}-8 & -7\\5 & 4\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix} = \begin{bmatrix}-25\\16\end{bmatrix}_S$$