DM841

Discrete Optimization

Part I

Lecture 14

Further Notions of Local Consistency

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Outline

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1. Higher Order Consistencies

2. Weaker arc consistencies

Higher Order Consistencies

- arc consistency does not remove all inconsistencies: even if a CSP is arc consistent there might be no solution
- arc consistency deals with each constraint separately
- stronger consistencies techniques are studied:
  - path consistency (generalizes arc consistency to arbitrary binary constraints)
  - restricted path consistency
  - k-consistency
  - (i, j)-consistent

Path consistency

Given $\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \rangle$ normalized:

- Given two variables x_i, x_j, the pair $(v_i, v_j) \in D(x_i) \times D(x_j)$ is p-path consistent iff forall $Y = (x_i = x_{k_1}, x_{k_2}, \ldots, x_{k_p} = x_j)$ with $C_{k_q, k_{q+1}} \in \mathcal{C}$ $\exists \tau : \tau[Y] = (v_i = v_{k_1}, \ldots, v_{k_p} = v_j) \in \pi_Y(\mathcal{D})$ and $(v_{k_q}, v_{k_{q+1}}) \in C_{k_q, k_{q+1}}$, $q = 1, \ldots, p-1$
- the CSP $\mathcal{P}$ is p-path consistent iff for any $(x_i, x_j), i \neq j$ any locally consistent pair of values (ie, satisfying all binary constraints between x_i, x_j) is p-path consistent.

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3), D(x_i) = \{1, 2\}, \mathcal{C} \equiv \{x_1 \neq x_2, x_2 \neq x_3\} \rangle$$

Not path consistent: e.g., for $(x_1, 1), (x_3, 2)$ there is no x_2
$\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \cup \{x_1 = x_3\} \rangle$ is path consistent (local consistency of x_1, x_3 removes values $x_1 \neq x_3$)

Alternative definition:

- constraint composition:
  $C_{x_1,x_3} = C_{x_1,x_2} \cdot C_{x_2,x_3} = \{(a,b) \mid \exists c, (a,c) \in C_{x_1,x_2}, (c,b) \in C_{x_2,x_3})\}$
- A normalized CSP $\mathcal{P}$ is 2-path consistent if for each subset $\{x_1, x_2, x_3\}$ of its variables we have $C_{x_1,x_3} \subseteq C_{x_1 x_2} \cdot C_{x_2 x_3}$
- Note: the sequence is arbitrary and the order irrelevant hence 6 conditions needs to be considered
- A CSP without binary constraints is trivially path consistent

Path Consistency rule 1 (propagator):

$$\frac{\langle C_{xy}, C_{xz}, C_{yz}; x \in D(x), y \in D(y), z \in D(z)\rangle}{\langle C'_{xy}, C_{xz}, C_{yz}; x \in D(x), y \in D(y), z \in D(z)\rangle}$$

where $C'_{xy} := C_{xy} \cap C_{xz} \cdot C_{zy}$
Path Consistency rule 2 (propagator):

$$\frac{\langle C_{xy}, C_{xz}, C_{yz}; x \in D(x), y \in D(y), z \in D(z)\rangle}{\langle C_{xy}, C'_{xz}, C_{yz}; x \in D(x), y \in D(y), z \in D(z)\rangle}$$

where $C'_{xz} := C_{xz} \cap C_{xy} \cdot C_{yz}$
Path Consistency rule 3 (propagator):

$$\frac{\langle C_{xy}, C_{xz}, C_{yz}; x \in D(x), y \in D(y), z \in D(z)\rangle}{\langle C_{xy}, C_{xz}, C'_{yz}; x \in D(x), y \in D(y), z \in D(z)\rangle}$$

where $C'_{yz} := C_{yz} \cap C_{yx} \cdot C_{xz}$

Example

$$\langle x < y, y < z, x < z; x \in [0..4], y \in [1..5], z \in [6..10] \rangle$$

is path consistent. Indeed:

$$C_{x,z} = \{(a,c) \mid a < c, a \in [0..4], c \in [6..10]\}$$
$$C_{x,y} = \{(a,b) \mid a < b, a \in [0..4], b \in [1..5]\}$$
$$C_{y,z} = \{(b,c) \mid b < c, b \in [1..5], c \in [6..10]\}$$

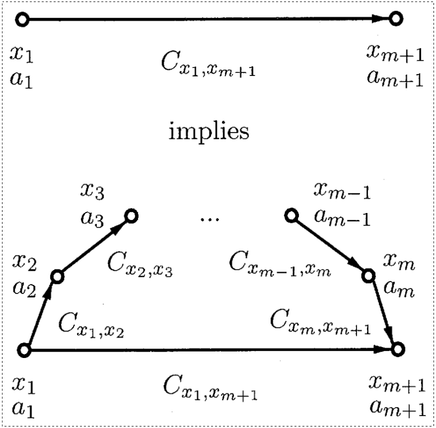
Example

$$\langle x < y, y < z, x < z; x \in [0..4], y \in [1..5], z \in [5..10] \rangle$$

is not path consistent. Indeed:
$C_{x,z} = \{(a,c) \mid a < c, a \in [0..4], c \in [5..10]\}$ and for $4 \in [0..4]$ and $5 \in [5..10]$ no $b \in [1..5]$ such that $4 < b$ and $b < 5$.

p-path consistency

The p-path consistency defined earlier generalizes 2-path consistency:



2-path consistency if the path has length 2

- CSP is p-path consistent $\iff$ 2-path consistent (Montanari, 1974). Proof by induction.
- Hence, sufficient to enforce consistency on paths of length 2.
- path consistency algorithms work with path of length two only and, like AC algorithms, make these paths consistent with revisions.
- Even if PC eliminates more inconsistencies than AC, seldom used in practice because of efficiency issues
- PC requires extensional representation of constraints and hence huge amount memory.
- Restricted PC does AC and PC only when a variable is left with one value.

k-consistency

Given $\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \rangle$, and set of variables $Y \subseteq X$ with $|Y| = k - 1$:

- a locally consistent instantiation I on Y is k-consistent iff for any kth variable $x_{i_k} \in X \setminus Y$ $\exists$ a value $v_{i_k} \in D(x_{i_k}) : I \cup \{x_{i_k}, v_{i_k}\}$ is locally consistent
- the CSP $\mathcal{P}$ is k-consistent iff for all Y of $k - 1$ variables any locally consistent I on Y is k-consistent.

Example

In general CSP, arc-consistent $\neq$ 2-consistent

$$D(x_1) = D(x_2) = \{1, 2, 3\}, \qquad x_1 \leq x_2, x_1 \neq x_2$$

arc consistent, every value has a support on one constraint
not 2-consistent, $x_1 = 3$ cannot be extended to x_2 and $x_2 = 1$ not to x_1 with both constraints
arc consistency: each binary constraint separately taken is not violated
2-consistency: any constraint is not violated

Example

$$D(x_i) = \{1, 2\}, i = 1, 2, 3; \mathcal{C} = \{(1, 1, 1), (2, 2, 2)\}$$

is $\mathcal{P}$ path consistent?
Yes because no binary constraint such that $X(C) \subseteq Y$
is $\mathcal{P}$ 3-consistent? No, because $(x_1, 1), (x_2, 2)$ is locally consistent but cannot be extended consistently to x_3.

Example

$$\langle D(x) = [1..2], D(y) = [1..2], D(z) = [2..4]; \mathcal{C} = \{x \neq y, x + y = z\}\rangle$$

- 1-consistent?
- 2-consistent?
- 3-consistent?

- A node consistent normalized CSP is arc consistent iff it is 2-consistent
- A node consistent normalized binary CSP is path consistent iff it is 3-consistent

That is, if the CSP is normalized:

- node consistency corresponds to 1-consistency
- arc consistency corresponds to 2-consistency
- path consistency corresponds to 3-consistency

However, in general CSP, no relationship between k-consistency and l-consistency for $k \neq l$ exists:

- for any $k > 1$, there exists an inconsistent CSP on k variables that is $(k-1)$-consistent but not k-consistent
- for any $k > 2$, there exists a consistent CSP on k variables that is not $(k-1)$-consistent but is k-consistent
- for any $k > 2$, there exists an inconsistent CSP on k variables that is k-consistent
- for any $k > 2$, there exists a consistent CSP on k variables that is not k-consistent

Example

- $\langle x_1 \neq x_2, x_2 \neq x_3, x_1 \neq x_3; x_1 \in \{0,1\}, x_2 \in \{0,1\}, x_3 \in \{0,1\}\rangle$
- $\langle x_1 \neq x_2, x_1 \neq x_3; x_1 \in \{a,b\}, x_2 \in \{a\}, ..., x_k \in \{a\}\rangle$ every $(k-1)$-consistent instantiation is a restriction of the consistent instantiation $(b, a, a, \ldots, a)$
- $\langle x_1 \neq x_2, x_2 \neq x_3, x_1 \neq x_3; x_1 \in \{1\}, x_2 \in \{1\}, x_3 \in \{1\}\rangle$
- $\langle x_1 \neq x_2, x_2 \neq x_3, x_1 \neq x_3; x_1 \in \{1\}, x_2 \in \{1,2,3\}, x_3 \in \{1,2,3\}\rangle$

- $\mathcal{P}$ is strongly k-consistent iff it is j-consistent $\forall j \leq k$
- constructing one requires $O(n^k d^k)$ time and $O(n^{k-1} d^{k-1})$ space.
- if $\mathcal{P}$ is strongly n-consistent then it is globally consistent
- (i, j)-consistent: any consistent instantiation of i different variables can be extended to a consistent instantiation including any j additional variables
  k consistency $\equiv (k-1, 1)$ consistent
- strongly (i, j)-consistent

Outline

Weaker arc consistencies

- reduce calls to Revise in coarse-grained algorithms (Forward Checking)
- reduce amount of work of Revise (Bound consistency)

Directional Arc Consistency

- Uses some linear ordering on the considered variables.
- Requires existence of supports only 'in one direction'
- A binary CSP $\mathcal{P}$ is directionally arc consistent (DAC) according to ordering $o = (x_1, \ldots, x_{k_n})$ on X, where $(k_1, \ldots, k_n)$ is a permutation of $(1, \ldots, n)$ iff for all $C_{x_i,x_j} \in \mathcal{C}$, if $x_i <_o x_j$ then x_i is arc consistent on C_{x_i,x_j}.
- CSP is dir. arc consistent if it is closed under application of arc consistency rule 1.

Example

$$\langle x < y; x \in [2..10], y \in [3..7] \rangle$$

not arc consistent but directionally arc consistent for the order (y, x)

Forward checking

Given $\mathcal{P}$ binary and $Y \subseteq X \ : |D(x_i)| = 1 \forall x_i \in Y$:

- $\mathcal{P}$ is forward checking consistent according to instantiation I on Y iff it is locally consistent and for all $x_i \in Y$, for all $x_j \in X \setminus Y$ for all $C(x_i, x_j) \in \mathcal{C}$ is arc consistent on $C(x_i, x_j)$.

  (all constraints between assigned and not assigned variables are consistent.)

- $O(ed)$ time (Revise called only once per arc)
- Extension to non-binary constraints
- Example:

  $\langle D(x_1) = D(x_2) = [1..5], D(x_3) = [1..3]; \mathcal{C} = \{x_1 < x_2, x_2 = x_3, x_1 > x_3\}\rangle$

  after $x_1 = 3$

Other Lookahead Filtering

Defined only by procedure, not by fixed point definition

Algorithm partial lookahead and full lookahead

procedure $PL(N, Y, x_i)$;
$FC(N, Y, x_i)$;
foreach $j \leftarrow i + 1$ ***to*** n **do**
  foreach $k \leftarrow j + 1$ ***to*** $n \mid c_{jk} \in C_N$ **do**
    if ***not*** $\texttt{Revise}(x_j, c_{jk})$ **then** return false

procedure $FL(N, Y, x_i)$;
$FC(N, Y, x_i)$;
foreach $j \leftarrow i + 1$ ***to*** n **do**
  foreach $k \leftarrow i + 1$ ***to*** $n, k \neq j \mid c_{jk} \in C_N$ **do**
    if ***not*** $\texttt{Revise}(x_j, c_{jk})$ **then** return false

Bound consistency

- domains inherit total ordering on $\mathbf{Z}$,
  $\min_D(x)$ and $\max_D(x)$ called bounds of $D(x)$
- Given $\mathcal{P}$ and C,
  a bounded support τ on C is a tuple that satisfies C and such that for all $x_i \in X(C)$, $\min_D(x_i) \leq \tau[x_i] \leq \max_D(x_i)$,
  that is, $\tau \in C \cap \pi_{X(C)}(D^I)$ (instead of D)

$$D^I(x_i) = \{v \in \mathbf{Z} \mid \min_D(x_i) \leq v \leq \max_D(x_i)\}$$

- C is bound($\mathbf{Z}$) consistent iff $\forall x_i \in X$ its bounds belong to a bounded support on C
- C is range consistent iff $\forall x_i \in X$ all its values belong to a bounded support on C
- C is bound($\mathbf{D}$) consistent iff $\forall x_i \in X$ its bounds belong to a support on C

- $GAC < (\text{bound}(\mathbf{D}), \text{range}) < \text{bound}(\mathbf{Z})$ (strictly stronger)
  bound(**D**) and range are incomparable
- most of the time gain in efficiency

Example

$$\text{sum}(x_1, \ldots, x_k, k)$$

GAC is NP-complete (reduction from SubSet problem).
But bound(**Z**) is polynomial: test $\forall 1 \leq i \leq n$:
$\min_D(x_i) \geq k - \sum_{j \neq i} \max_D(x_j)$
$\max_D(x_i) \leq k - \sum_{j \neq i} \min_D(x_j)$

References