DM841

Discrete Optimization

Part I

Filtering algorithms for global constraints

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Outline

1. Global Constraints
   - Scheduling
2. Soft Constraints
3. Optimization Constraints

Outline

1. Global Constraints
    Scheduling

2. Soft Constraints

3. Optimization Constraints

Declarative and Operational Semantic

- Declarative Semantic: specify **what** the constraint means. Evaluation criteria is expressivity.
- Operational Semantic: specify **how** the constraint is computed, i.e., is kept *consistent* with its declarative semantic. Evaluation criteria are efficiency and effectiveness.

Example

So far, we have defined only the Declarative Semantic of the `alldifferent` constraint, not its Operational Semantic.

Domain Consistency

Definition

A constraint C on the variables $x_1, \ldots, x_r$ with respective domains $D_1, \ldots, D_r$ is called domain consistent (or generalized/hyper-arc consistent) if for each variable x_i and each value $d_i \in D_i$ there exist compatible values in the domains of all the other variables of C, that is, there exists a tuple $(d_1, \ldots, d_i, \ldots, d_r) \in C$.

In other terms: If value v is in the domain of variable x, then there exists a solution to the constraint with value v assigned to variable x.

Examples: `alldifferent` (`distinct`), knapsack, ...

Definition

Filtering algorithm $\equiv$ reduction rule: reduce $D(x_i)$ for $1 \leq i \leq r$ such that it still contains all values that the variable can assume in a solution of C.

$D(x_i) \leftarrow D(x_i) \cap \{d_i \in D(x_i) | D(x_1 \times D(x_{i-1}) \times \{v_i\} \times D(x_{i+1}) \times \ldots, D(x_r)\} \cap C \neq \emptyset\}$
Generic arc consistency algorithms are in $O(erd^r)$.

Consistency and Filtering Algorithms

- Different filtering algorithms, which must be able to:
  1. Check consistency of C w.r.t. the current variable domains
  2. Remove inconsistent values from the variable domains

- The stronger is the level of consistency, the higher is the complexity of the filtering algorithm: Different level of consistency (domain, bound(Z), bound(D), range, value):
  - complete filtering, optimal pruning, domain completeness $\equiv$ domain/arc consistency
  - partial filtering, bound completeness $\equiv$ bound relaxed completeness

... again the `alldifferent` case

There exist in literature several filtering algorithms for the `alldifferent` constraints.

Decomposition Approach

A decomposition of a global constraint C is a polynomial time transformation $\delta_k(\mathcal{P})$ replacing C by some new bounded arity constraint (and possibly new variables) while preserving the set of tuples allowed on $X(C)$.

Global Constraint Decomposition

Given any $\mathcal{P} = \langle X(C), \mathcal{D}, \mathcal{C} = \{C\}\rangle$, $\delta_k(\mathcal{P})$ is such that

- $X(C) \subseteq X_{\delta_k(\mathcal{P})}$
- for all $x_i \in X(C)$, $D(x_i) = D_{\delta_k(\mathcal{P})}(x_i)$
- for all $C_j \in C_{\delta_k(\mathcal{P})}$, $|X(C_j)| \leq k$ and
- $sol(\mathcal{P}) = \pi_{X(C)}(sol(\delta_k(\mathcal{P}))$

Example

atmost$(x_1, \ldots, x_n, p, v)$ (at most p variables in $x_1, \ldots, x_n$ take value v).
Decomposition: $n+1$ additional variables $y_0, \ldots, y_n$
$(x_i = v \wedge y_i = y_{i-1} + 1) \vee (x_i \neq v \wedge y_i = y_{i-1})$ for all i, $1 \leq i \leq n$, and domains $D(y_0) = \{0\}$ and $D(y_i) = \{0, \ldots, p\}$ for $1 \leq i \leq n$.

These decompositions can be:

- preserving solutions
- preserving generalized arc consistency
- preserving the complexity of enforcing generalized arc consistency

The decomposition of `atmost` preserves solutions and generalized arc consistency
For the `alldifferent` only preserving solutions. Yet sometimes it is possible to construct a specialized algorithm that enforces GAC in polynomial time.

alldifferent

alldifferent constraint

Let $x_1, x_2, \ldots, x_n$ be variables. Then:

$$\texttt{alldifferent}(x_1, \ldots, x_n) = \{(d_1, \ldots, d_n) \mid \forall i \; d_i \in D(x_i), \quad \forall i \neq j, \; d_i \neq d_j\}.$$

Complete Filtering for alldifferent

1. build value graph $G = (X, D(X), E)$
2. compute maximum matching M in G
3. **if** $|M| < |X|$ **then return false**
4. mark all arcs in oriented graph G_M that are not in M as unused
5. compute SCCs in G_M and mark all arcs in a SCC as used
6. perform breadth-first search in G_M starting from M-free vertices, and mark all traversed arcs as used if they belong to an even path
7. **for all** arcs (x_i, d) in G_M marked as unused **do**
   $D(x_i) := D(x_i) \setminus d$
   if $D(x_i) = \emptyset$ **then return false**
8. **return true**

Overall complexity: $O(n\sqrt{m} + (n + m) + m)$
It can be updated incrementally if other constraints remove some values.

Example

$\text{alldiff}(x_1, \ldots, x_5)$

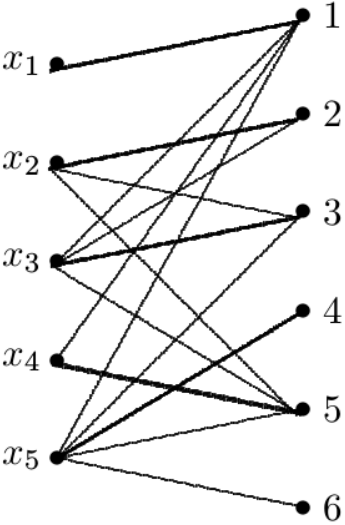
$D_{x_1} = \{1\}$

$D_{x_2} = \{2, 3, 5\}$

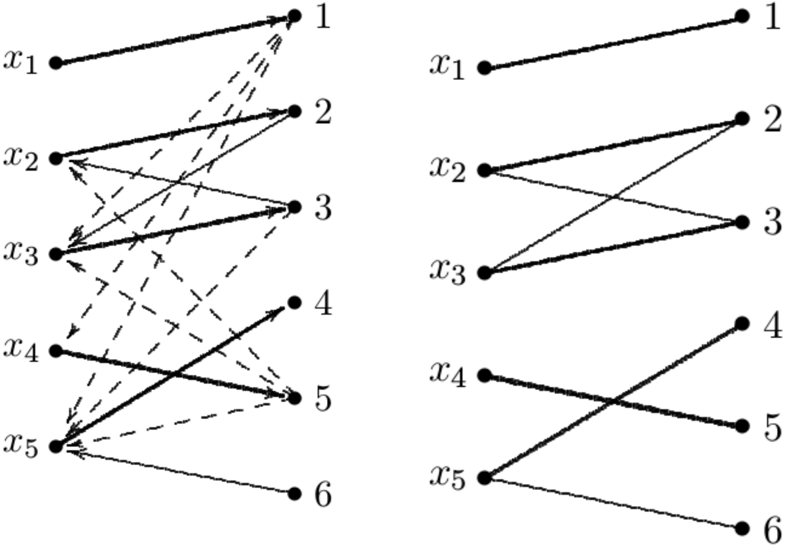
$D_{x_3} = \{1, 2, 3, 5\}$

$D_{x_4} = \{1, 5\}$

$D_{x_5} = \{1, 3, 4, 5, 6\}$



Example



Relaxed Consistency

Definition

A constraint C on the variables $x_1, \dots, x_m$ with respective domains $D_1, \dots, D_m$ is called bound(Z) consistent if for each variable x_i and each value $d_i \in \{\min(D_i), \max(D_i)\}$ there exist compatible values between the min and max domain of all the other variables of C, that is, there exists a value $d_j \in [\min(D_i), \max(D_i)]$ for all $j \neq i$ such that $(d_1, \dots, d_i, \dots, d_k) \in C$.

Definition

A constraint C on the variables $x_1, \dots, x_m$ with respective domains $D_1, \dots, D_m$ is called range consistent if for each variable x_i and each value $d_i \in D_i$ there exist compatible values between the min and max domain of all the other variables of C, that is, there exists a value $d_j \in [\min(D_i), \max(D_i)]$ for all $j \neq i$ such that $(d_1, \dots, d_i, \dots, d_k) \in C$.

($\text{bound}(\mathbf{D})$ if its bounds belong to a support on C)
$GAC < (\text{bound}(\mathbf{D}), \text{range}) < \text{bound}(\mathbf{Z})$

Bound Consistency [Mehlorn&Thiel2000]

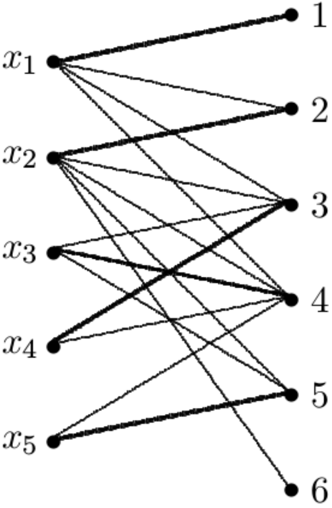
Definition (Convex Graph)

A bipartite graph $G = (X, Y, E)$ is convex if the vertices of Y can be assigned distinct integers from $[1, |Y|]$ such that for every vertex $x \in X$, the numbers assigned to its neighbors form a subinterval of $[1, |Y|]$.

In convex graph we can find a matching in linear time.

Example

$$
\begin{aligned}
D_{x_1} &= \{1, 2, \quad 4 \quad\} \\
D_{x_2} &= \{\quad 2, 3, \quad\quad 6\} \\
D_{x_3} &= \{\quad\quad 3, \quad 5 \quad\} \\
D_{x_4} &= \{\quad\quad 3, 4 \quad\quad\} \\
D_{x_5} &= \{\quad\quad\quad 4, 5 \quad\}
\end{aligned}
\qquad\qquad
\begin{aligned}
I_{x_1} &= \{1, 2, 3, 4 \quad\quad\} \\
I_{x_2} &= \{\quad 2, 3, 4, 5, 6\} \\
I_{x_3} &= \{\quad\quad 3, 4, 5 \quad\} \\
I_{x_4} &= \{\quad\quad 3, 4 \quad\quad\} \\
I_{x_5} &= \{\quad\quad\quad 4, 5 \quad\}
\end{aligned}
$$



Survey of complexity: effectiveness and efficiency

Consistency	Idea	Complexity	Amort.	Reference(s)
arc		$O(n^2)$		[VanHentenryck1989]
bound	Hall	$O(n \log n)$		[Puget1998]
	Flows			[Mehlhorn&Thiel2000]
	Hall			[Lopez&All2003]
		$O(n)$		[Mehlhorn&Thiel2000]
				[Lopez&All2003]
range	Hall	$O(n^2)$		[Leconte1996]
domain	Flows	$O(n\sqrt{m})$	$O(n\sqrt{k})$	[Régin1994],[Costa1994]

Where $n =$ number of variables, $m = \sum_{i \in 1 \dots n} |D_i|$, and $k =$ number of values removed.

Filtering cardinality

cardinality or gcc (global cardinality constraint)

Let $x_1, \ldots, x_n$ be assignment variables whose domains are contained in $\{v_1, \ldots, v_{n'}\}$ and let $\{c_{v_1}, \ldots, c_{v_{n'}}\}$ be count variables whose domains are sets of integers. Then

$$\texttt{cardinality}([x_1, ..., x_n],[c_{v_1}, ..., c_{v_{n'}}]) = \{(w_1, ..., w_n, o_1, ..., o_{n'}) \mid w_j \in D(x_j)\,\forall j, \; \texttt{occ}(v_i, (w_1, ..., w_n)) = o_i \in D(c_{v_i})\,\forall i\}.$$

(occ number of occurrences)

$\rightsquigarrow$ generalization of alldifferent
NP-hard to filter domain of all variables. But if constant intervals, then polynomial algorithm via network flows. (integral feasible (s, t)-flow)

Filtering knapsack

Knapsack and Sum constraints (Linear constraints over integer variables)

Let $x_1, \ldots, x_n, z, c$ be integer variables:

$$\text{knapsack}([x_1, \ldots, x_n], z, c) =$$

$$\left\{(d_1, \ldots, d_n, d) \mid d_i \in D(x_i)\,\forall i, d \in D(z), d \leq \sum_{i=1,\ldots,n} c_i d_i\right\} \cap$$

$$\left\{(d_1, \ldots, d_n, d) \mid d_i \in D(x_i)\,\forall i, d \in D(z), d \geq \sum_{i=1,\ldots,n} c_i d_i\right\}.$$

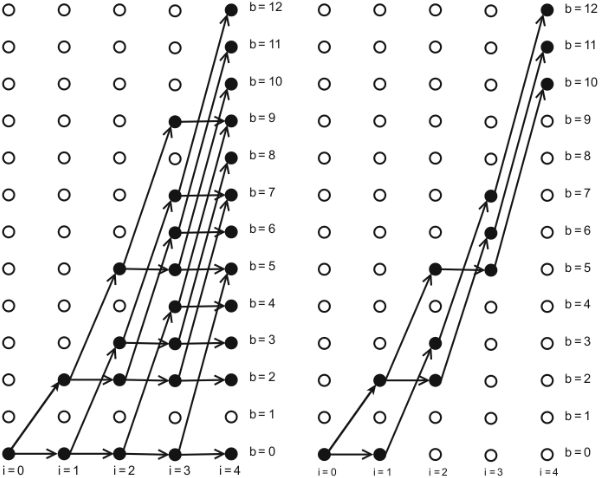
Binary Knapsack (Linear constraints over Boolean variables)

$\sum c_i x_i = z, x_i \in \{0, 1\} \rightsquigarrow l_z \leq \sum c_i x_i \leq u_z$

Variant of the subset sum problem: Given a set of numbers find a subset whose sum is 0.

Eg: $-7, -3, -2, 5, 8 \rightsquigarrow -3 - 2 + 5 = 0$

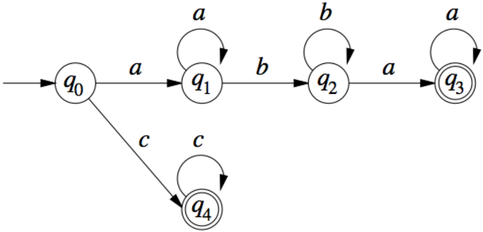
$10 \leq 2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 12$

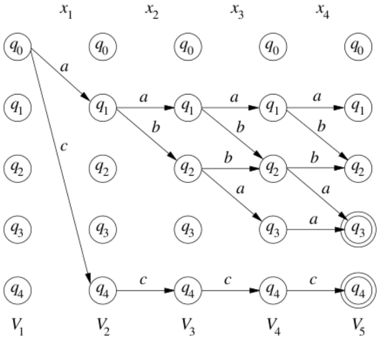


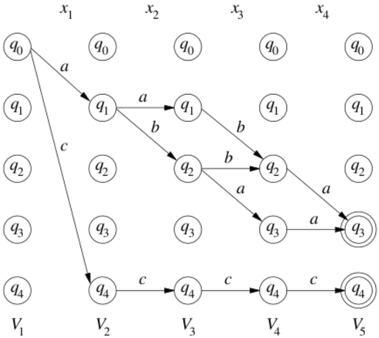
Filtering regular

"regular" constraint

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of variables with $D(x_i) \subseteq \Sigma$ for $1 \leq i \leq n$. Then
$\texttt{regular}(X, M) = \{(d_1, ..., d_n) \mid \forall i, d_i \in D(x_i), [d_1, d_2, \ldots, d_n] \in L(M)\}$.




x_1 x_2 x_3 x_4
q_0 q_0 q_0 q_0 q_0
q_1 q_1 q_1 q_1 q_1
q_2 q_2 q_2 q_2 q_2
q_3 q_3 q_3 q_3 q_3
q_4 q_4 q_4 q_4 q_4
a b c
V_1 V_2 V_3 V_4 V_5


x_1 x_2 x_3 x_4
q_0 q_0 q_0 q_0 q_0
q_1 q_1 q_1 q_1 q_1
q_2 q_2 q_2 q_2 q_2
q_3 q_3 q_3 q_3 q_3
q_4 q_4 q_4 q_4 q_4
a b c
V_1 V_2 V_3 V_4 V_5

Other Filtering Algorithms

- `linear`
- `element`
- `disjunctive`
- `cumulative`

linear

$$\sum_{i=1}^{n} a_i x_i + b \begin{matrix} \leq \\ = \\ \geq \end{matrix} 0 \qquad x_i \in [l_i, h_i]$$

Example

$$3x + 4y - 5z \leq 7$$

$$x \leq \frac{7 - 4y + 5z}{3} \quad \Longrightarrow \quad x \leq \left\lfloor \frac{7 - 4l_y + 5h_z}{3} \right\rfloor$$

$$[l_x, h_x] \longleftarrow \left[l_x, \min(h_x, \left\lfloor \frac{7 - 4l_y + 5h_z}{3} \right\rfloor) \right]$$

$$\sum_{i \in POS} a_i x_i - \sum_{i \in NEG} a_i x_i \leq b$$

$$x_{\leq} \frac{b - 4y + 5z}{3} \quad \Longrightarrow \quad x_j \leq \frac{b - \sum_{i \in POS \setminus \{j\}} a_i x_i + \sum_{i \in NEG} a_i x_i}{a_j}$$

$$\alpha_j = \frac{b - \sum_{i \in POS \setminus \{j\}} a_i l_i + \sum_{i \in NEG} a_i h_i}{a_j}$$

$$\beta_j = \frac{b - \sum_{i \in POS \setminus \{j\}} a_i h_i + \sum_{i \in NEG} a_i l_i}{a_j}$$

$$[l_j, h_j] \longleftarrow [\max(l_x, \lceil \beta_j \rceil), \min(h_j, \lfloor \alpha_j \rfloor)]$$

(domain consistency is NP-complete, this one is bound(Z))

element

- element$(y, \vec{a}, z) \equiv z = a_y$

$$D(z) \longleftarrow D(z) \cap \{a_i \mid i \in D(y)\}$$
$$D(y) \longleftarrow \{i \in D(y) \mid a_i \in D(z)\}$$

- element$(y, \vec{x}, z) \equiv z = x_y$

$$D(z) \longleftarrow D(z) \cap \bigcup_{i \in D(y)} D_{x_i}$$
$$D(y) \longleftarrow \{i \in D(y) \mid D(z) \cap D_{x_i} = \emptyset\}$$
$$D(x_i) \longleftarrow \begin{cases} D(z) & \textit{if } D(y) = \{i\} \\ D(x_i) & \textit{else} \end{cases}$$

Outline

Edge Finding

$$\text{If } L_J - E_{J\cup\{i\}} < p_i + p_J, \text{ then } i \gg J \quad (a)$$
$$\text{If } L_{J\cup\{i\}} - E_J < p_i + p_J, \text{ then } i \ll J \quad (b)$$

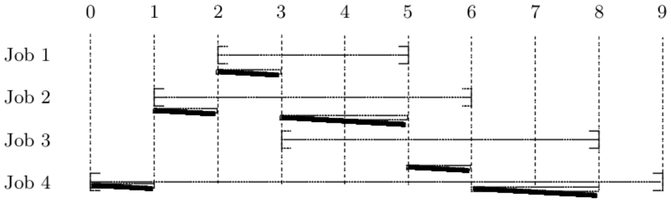
$$\text{If } i \gg J, \text{ then update } E_i \text{ to } \max\left\{E_i, \max_{J' \subset J}\{E_{J'} + p_{J'}\}\right\}.$$

$$\text{If } i \ll J, \text{ then update } L_i \text{ to } \min\left\{L_i, \min_{J' \subset J}\{L_{J'} - p_{J'}\}\right\}.$$

j	p_j	E_j	L_j
1	1	2	5
2	3	1	6
3	1	3	8
4	3	0	9

$O(n^2)$ algorithm

j	p_j	E_j	L_j
1	1	2	5
2	3	1	6
3	1	3	8
4	3	0	9



i	J_i	$\bar{p}$	k	J_{ik}	$L_k - E_i$	$p_i + \bar{p}_{J_{ik}}$
1	$\{1,2,3,4\}$	$(1,2,1,2)$	4	$\{2,3,4\}$	$9-2$	$1+5$
			3	$\{2,3\}$	$8-2$	$1+3$
			2	$\{2\}$	$6-2$	$1+3$
2	$\{1,2,3,4\}$	$(1,3,1,2)$	4	$\{1,3,4\}$	$9-1$	$3+4$
			3	$\{1,3\}$	$8-1$	$3+2$
			1	$\{3\}$	$5-1$	$3+1$
3	$\{2,3,4\}$	$(0,2,1,2)$	4	$\{2,4\}$	$9-3$	$1+4$
			2	$\{2\}$	$6-3$	$1+2$
4	$\{1,2,3,4\}$	$(1,3,1,3)$	3	$\{1,2,3\}$	$8-0$	$3+5$
			2	$\{1,2\}$	$6-0$	$3+4$

Conclude that $4 \gg \{1,2\}$ and update E_4 from 0 to 5

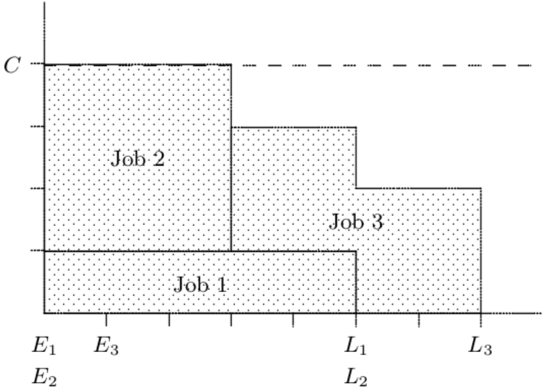
Not first, Not Last

$$\text{If } L_J - E_i < p_i + p_J, \text{ then } \neg(i \ll J). \quad (a)$$
$$\text{If } L_i - E_J < p_i + p_J, \text{ then } \neg(i \gg J). \quad (b)$$

$$\text{If } \neg(i \ll J), \text{ then update } E_i \text{ to } \max\left\{E_i, \min_{j\in J}\{E_j + p_j\}\right\} \quad (a)$$
$$\text{If } \neg(i \gg J), \text{ then update } L_i \text{ to } \min\left\{L_i, \max_{j\in J}\{L_j - p_j\}\right\} \quad (b)$$

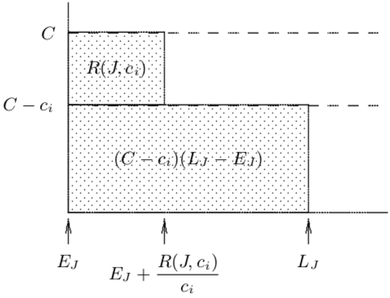
Cumulative Scheduling

j	p_j	c_j	E_j	L_j
1	5	1	0	5
2	3	3	0	5
3	4	2	1	7



Edge Finding

$$\text{If } e_i + e_J > C \cdot (L_J - E_{J \cup \{i\}}), \text{ then } i > J. \quad (a)$$
$$\text{If } e_i + e_J > C \cdot (L_{J \cup \{i\}} - E_J), \text{ then } i < J. \quad (b)$$



$$\text{If } i > J \text{ and } R(J, c_i) > 0, \text{ update } E_i \text{ to } \max\left\{E_i, E_J + \frac{R(J, c_i)}{c_i}\right\}.$$
$$\text{If } i < J \text{ and } R(J, c_i) > 0, \text{ update } L_i \text{ to } \min\left\{L_i, L_J - \frac{R(J, c_i)}{c_i}\right\}.$$

Filtering Algorithm Design

1. Filtering algorithms based on a generic algorithm
   Simple AC algorithms. Eg, `element`:

   $$\texttt{element}(y, [2, 4, 8, 16, 32], x), x \in \{1, 2, 3, 4, 5\}$$

2. Filtering algorithms based on existing algorithms
   Reuse existing algorithms for filtering (e.g., flows algorithms, dynamic programming).

3. Filtering algorithms based on ad-hoc algorithms
   Pay particular attention to incrementality and amortized complexity

4. Filtering algorithms based on model reformulation
   See the Constraint Decomposition approach

Outline

Soft Constraints

Soft constraint

A *soft constraint* is a constraint that may be violated. We measure the violation of each constraint, and the goal is to minimize the total amount of violation of all soft-constraints.

Definition

A *violation measure* for a soft-constraint $C(x_1, \ldots, x_n)$ is a function

$$\mu : D(x_1) \times \cdots \times D(x_n) \to \mathbb{Q}.$$

This measure is represented by a *cost* variable z.

Violation measures

- The variable-based violation measure μ_{var} counts the minimum number of variables that need to change their value in order to satisfy the constraint.
- The decomposition-based violation measure μ_{dec} counts the number of constraints in the binary decomposition that are violated.

The soft-alldifferent

Definition

Let $x_1, x_2, \ldots, x_n, z$ be variables with respective finite domains $D(x_1), D(x_2), \ldots, D(x_n), D(z)$. Let μ be a violation measure for the alldifferent constraint. Then

$$\text{soft-alldifferent}(x_1, \ldots, x_n, z, \mu) = \{(d_1, \ldots, d_n, d) \mid \forall i. d_i \in D(x_i), d \in D(z), \mu(d_1, \ldots, d_n) \leq d\}$$

is the soft alldifferent constraint with respect to μ.

The soft-alldifferent: an example

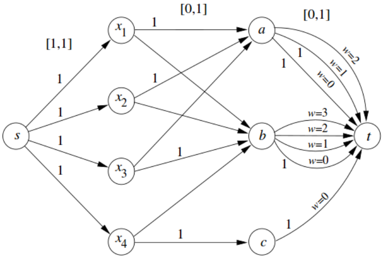
Example

Consider the following CSP

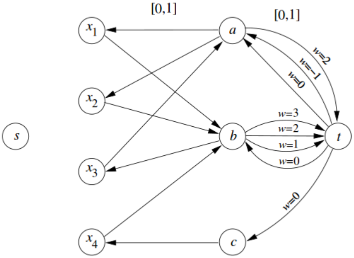
$$x_1 \in \{a, b\}, x_2 \in \{a, b\}, x_3 \in \{a, b\}, x_4 \in \{a, b, c\}, z \in \mathbb{Z}^+$$
$$\text{soft-alldifferent}(x_1, x_2, x_3, x_4, \mu, z)$$
$$\min z$$

We have for instance $\mu_{var}(b, b, b, b) = 3$ and $\mu_{dec}(b, b, b, b) = 6$.

Filtering of soft-alldiff



Flow network and feasible flow



Residual graph

Outline

1. Global Constraints
   Scheduling
2. Soft Constraints
3. Optimization Constraints

Optimization Constraints

Optimization Constraint bring the costs of variable-value pair into the declarative semantic of the constraints.

The filtering does take into account the cost, and a tuple may be inconsistent because it does not lead to a solution of “at least” a given cost.
Basic approach, solve a sequence of decision problems, allows one-way inference.
More powerful approach takes into account two-way inference.

gcc with costs

cardinality or cost_gcc (global cardinality constraint with costs)

Let $x_1, \ldots, x_n$ be assignment variables whose domains are contained in $\{v_1, \ldots, v_{n'}\}$ and let $\{c_{v_1}, \ldots, c_{v_{n'}}\}$ be count variables whose domains are sets of integers and $w(x, d) \in Q$ are costs. Then

$$\begin{aligned}\texttt{cost\_gcc}([x_1, ..., x_n], [c_{v_1}, ..., c_{v_{n'}}], z, w) = \\ \{(d_1, ..., d_n, o_1, ..., o_{n'}) \mid \\ \{(d_1, ..., d_n, o_1, ..., o_{n'}) \in \texttt{gcc}(([x_1, ..., x_n], [c_{v_1}, ..., c_{v_{n'}}]), \\ \forall d_j \in D(x_j)\, d \in D(z) \sum_i w(x_i, d_i) \leq d\}.\end{aligned}$$

Filtering for `cost_gcc`

(works on constant intervals)
Extend the (s,t)-network saw for `gcc` by weigths $w(x_i, v_i)\ \forall v_i$

1. compute initial min-cost feasible (s,t)-flow, f. $(O(n(m + n \log n)$
2. For an arc uv with $f(a) = 0$ compute min cost directed path P from v to u in the residual graph. $P + a$ is a directed circuit.
3. since f is integer we can rerout one unit in the circuit and obtain: $cost(f') = cost(f) + cost(P)$.
4. if $cost(f') > max(D(z))$ remove v from $D(x_i)$

2.-4. in $O(\Delta(m + n \log n))$

Reduced-Cost Based Filtering [Focacci&all1999]

Definition

Let $X = \{x_1, ..., x_n\}$ be a set of variables with corresponding finite domains $D(x_1), ..., D(x_n)$. We assume that each pair (x_i, j) with $j \in D(x_i)$ induces a cost c_{ij}.
We extend any global constraint C on X to an optimization constraint opt_C by introducing
a cost variable z (that we wish to minimize) and defining

$$\text{opt\_C}(x_1, ..., x_n, z, c) = \{(d_1, ..., d_n, d) | (d_1, ..., d_n) \in C(x_1, ..., x_n),$$

$$\forall i. d_i \in D(x_i), d \in D(z), \sum_{i=1,...,n} c_{id_i} \leq d\}.$$

Linear Relaxation

We introduce binary variables y_{ij} for all $i \in \{1, ..., n\}$ and $j \in D(x_i)$, such that

$$
\begin{aligned}
x_i = j \Leftrightarrow y_{ij} = 1, \qquad & \forall i = 1, \ldots, n, \forall j \in D(x_i), \\
x_i \neq j \Leftrightarrow y_{ij} = 0, \qquad & \forall i = 1, \ldots, n, \forall j \in D(x_i) \\
\sum_{j \in D(x_i)} y_{ij} = 1, \qquad & \forall i = 1, \ldots, n.
\end{aligned}
$$

+ constraint dependent linear inequalities

The reduced-costs are given w.r.t. the objective:

$$\sum_{i=1,\ldots,n} \sum_{j \in D(x_i)} c_{ij} y_{ij}$$

Example

alldiff

$$
\begin{array}{ll}
\min & \sum_{i,j} c_{i,j} y_{i,j} \\
 & \sum_{j \in D(x_i)} y_{ij} = 1, \quad \forall i = 1, \ldots, n \\
 & \sum_{i=1,\ldots,n} y_{ij} \leq 1, \quad \forall j \in D(x_i) \\
 & y_{ij} \geq 0
\end{array}
$$

Filtering by Reduced-Cost (aka “variable fixing”)

Recall that reduced-costs estimate the increase of the objective function when we force a variable into the solution.

Let $\bar{c}_{ij}$ be the reduced cost for the variable-value pair $x_i = j$, and let z^* be the optimal value of the current linear relaxation.

We apply the following filtering rule:

$$\textbf{if } z^* + \bar{c}_{ij} > \max D(z) \textbf{ then } D(x_i) \leftarrow D(x_i) \setminus \{j\}.$$

References

Algorithms from the paper discussed at the blackboard