DM841

Discrete Optimization

Part I

# Notions of Local Consistency

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# Goals

- Establish formalism around constraint handling
- Learning general constraint propagation algorithms

# Outline

1. Definitions

# Reasoning with Constraints



Constraint Propagation, related notions:

- constraint relaxation
- filtering algorithms
- narrowing algorithms
- constraint inference
- simplification algorithms
- label inference
- local consistency enforcing
- rules iteration
- proof rules

Local Consistency define properties that the constraint problem must satisfy after constraint propagation

Rules Iteration defines properties on the process of propagation itself, that is, kind and order of operations of reduction applied to the problem

# Notation and Terminology



Finite domains $\rightsquigarrow$ w.l.g. $D \subseteq \mathbf{Z}$

Constraint $C$: relation on a (ordered) *subsequence* of variables

- $X(C) = (x_{i_1}, \ldots, x_{i_{|X(C)|}})$ is the scheme or scope of C
- $|X(C)|$ is the arity of $C$ (unary/binary/non-binary)
- $C \subseteq \mathbf{Z}^{|X(C)|}$ containing combinations of valid values (or tuples) $\tau \in \mathbf{Z}^{|X(C)|}$
- constraint check: testing whether a $\tau$ satisfies $C$
- $\mathcal{C}$: a $t$-tuple of constraints $\mathcal{C} = (C_1, \ldots, C_t)$
- expression
  - extensional: specifies satisfying tuples (aka `table` or `extensional` via **`DFA`** or **`TupleSet`** in gecode).
    eg. $c(x_1, x_2) = \{(2,2),(2,3),(3,2),(3,3)\}$
  - intensional: specifies the characteristic function. eg. $\texttt{alldiff}(x_1, x_2, x_3)$

# CSP



Input:

- **Variables** $X = (x_1, \ldots, x_n)$
- **Domain Expression** $\mathcal{D} = \{x_1 \in D(x_1), \ldots, x_n \in D(x_n)\}$
- $\mathcal{C}$ finite set of constraints each on a subsequence $Y$ of $X$.
  $C \in \mathcal{C}$ on $Y = (y_1, \ldots, y_k)$ is $C \subseteq D(y_1) \times \ldots \times D(y_k)$

a constrained satisfaction problem (CSP) is

$$\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \rangle$$

$(v_1, \ldots, v_n) \in D(x_1) \times \ldots \times D(x_n)$ is a solution of $\mathcal{P}$
if for each constraint $C_i \in \mathcal{C}$ on $x_{i_1} \ldots, x_{i_m}$ it is

$$(v_{i_1}, \ldots, v_{i_m}) \in C_i$$

CSP normalized: iff two different constraints do not involve exactly the same vars
CSP binary iff for all $C_i \in \mathcal{C}, |X(C)| = 2$

# Notation and Terminology

Given a tuple $\tau$ on a sequence $Y$ of variables and $W \subseteq Y$,

- $\tau[W]$ is the restriction of $\tau$ to variables in $W$ (ordered accordingly)
- $\tau[x_i]$ is the value of $x_i$ in $\tau$
- if $X(C) = X(C')$ and $C \subseteq C'$ then for all $\tau \in C$ the reordering of $\tau$ according to $X(C')$ satisfies $C'$.

Example

$C(x_1, x_2, x_3): \quad x_1 + x_2 = x_3$

$C'(x_1, x_2, x_3): \quad x_1 + x_2 \leq x_3$

$C \subseteq C'$

# Notation and Terminology



- Given $Y \subseteq X(C)$, $\pi_Y(C)$ denotes the projection of $C$ on $Y$. It contains tuples on $Y$ that can be extended to a tuple on $X(C)$ satisfying $C$.
- given $X(C_1) = X(C_2)$, the intersection $C_1 \cap C_2$ contains the tuples $\tau$ that satisfy both $C_1$ and $C_2$
- join of $\{C_1 \dots C_k\}$ is the relation with scheme $\cup_{i=1}^{k} X(C_i)$ that contains tuples such that $\tau[X(C_i)] \in C_i$ for all $1 \leq i \leq k$.

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{D} = \{D(x_i) = \{1..5\}, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv \texttt{alldiff}(x_1, x_2, x_3), C_2 \equiv x_1 \leq x_2 \leq x_3, C_3 \equiv x_4 \geq 2x_2\}\rangle$$

$\pi_{x_1,x_2}(C_1) \equiv (x_1 \neq x_2)$
$C_1 \cap C_2 \equiv (x_1 < x_2 < x_3)$
join $\{C_1, \dots, C_3\} \equiv (x_1 < x_2 < x_3 \wedge x_4 \geq 2x_2)$

# Notation and Terminology



Given $\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \rangle$ the instantiation $I$ is a tuple on $Y = (x_1, \ldots, x_k) \subseteq X$: $((x_1, v_1), \ldots, (x_k, v_k))$

- $I$ on $Y$ is valid iff $\forall x_i \in Y$, $I[x_i] \in D(x_i)$
- $I$ on $Y$ is locally consistent iff it is valid and for all $C \in \mathcal{C}$ with $X(C) \subseteq Y$, $I[X(C)]$ satisfies $C$ (some constraints may have $X(C) \not\subseteq Y$)
- a solution to $\mathcal{P}$ is an instantiation $I$ on $X(\mathcal{C})$ which is locally consistent
- $I$ on $Y$ is globally consistent if it can be extended to a solution, i.e., there exists $s \in sol(\mathcal{P})$ with $I = s[Y]$

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{D} = \{D(x_i) = \{1..5\}, \forall i\},$$
$$\mathcal{C} = \{C_1 \equiv \texttt{alldiff}(x_1, x_2, x_3), C_2 \equiv x_1 \leq x_2 \leq x_3, C_3 \equiv x_4 \geq 2x_2\}\rangle$$

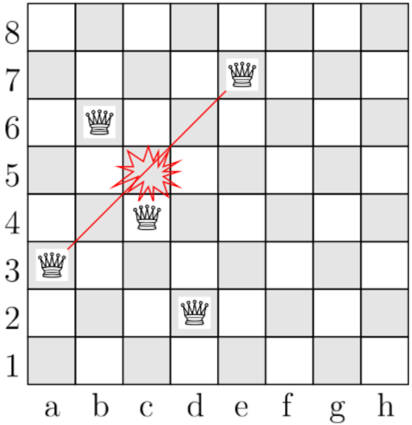
$\pi_{\{x_1, x_2\}}(C_1) \equiv (x_1 \neq x_2)$
$I_1 = ((x_1, 1), (x_2, 2), (x_4, 7))$ is not valid
$I_2 = ((x_1, 1), (x_2, 1), (x_4, 3))$ is local consistent since $C_3$ only one with $X(C_3) \subseteq Y$ and $I_2[X(C_3)]$ satisfies $C_3$
$I_2$ is not global consistent: $sol(\mathcal{P}) = \{(1, 2, 3, 4), (1, 2, 3, 5)\}$

# Notation and Terminology



- An instantiation $I$ on $\mathcal{P}$ is globally consistent if it can be extended to a solution of $\mathcal{P}$, globally inconsistent otherwise.
- A globally inconsistent instantiation is also called a (standard) nogood. (a partial instantiation that does not lead to a solution.)
- Remark: A locally inconsistent instantiation is a nogood. The reverse is not necessarily true

# Example

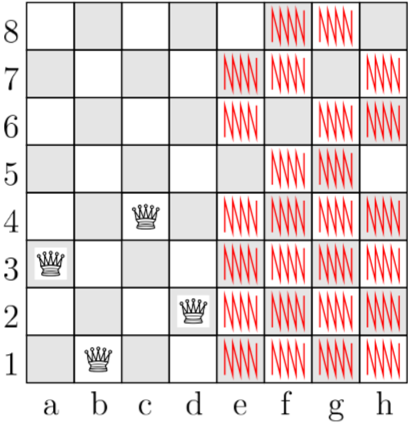



$\{(x_a,3),(x_b,6),(x_c,4),(x_d,2),(x_e,7)\}$ is locally inconsistent

☞ this is a nogood.

# Example






$\{(x_a,3),(x_b,1),(x_c,4),(x_d,2)\}$ is globally inconsistent

☞ this is a nogood.

# Notation and Terminology



CSP solved by extending partial instantiations to global consistent ones and backtracking at local inconsitencies $\rightsquigarrow$ is NP-complete!

Idea: make the problem more explicit (tighter)

$\mathcal{P}' \preceq \mathcal{P}$ iff $X_{\mathcal{P}'} = X_{\mathcal{P}}$ and any instantiation $I$ on $Y \subseteq X_{\mathcal{P}}$ locally inconsistent for $\mathcal{P}$ is locally inconsistent for $\mathcal{P}'$.

Example

$$\begin{aligned}\mathcal{P} =& \langle X = (x_1, x_2, x_3), \mathcal{D} = \{D(x_i) = [1..4], \forall i\},\\ &\quad \mathcal{C} = \{C_1 \equiv x_1 < x_2, C_2 \equiv x_2 < x_3,\\ &\quad C_3 \equiv \{(111), (123), (222), (333), (234)\}\}\rangle\\ \mathcal{P}' =& \langle X, \mathcal{D}, \mathcal{C}'\rangle, \mathcal{C}' = \{C_1, C_2, C_3' \equiv \{(123)\}\}\rangle\end{aligned}$$

$\mathcal{P}' \preceq \mathcal{P}$: All locally inconsitent instantiations on $Y \subseteq X_{\mathcal{P}}$ for $\mathcal{P}$ are locally inconsistent for $\mathcal{P}'$. Indeed $X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{D}_{\mathcal{P}'} = \mathcal{D}$ and $C_1 = C_1', C_2 = C_2', X(C_3) = X(C_3'), C_3' \subset C_3$.
However not all solutions are preserved!

Example

$\mathcal{P} = \langle X = (x_1, x_2, x_3), \mathcal{D} = \{D(x_i) = [1..4], \forall i\},$
$\mathcal{C} = \{C_1 \equiv x_1 < x_2, C_2 \equiv x_2 < x_3, C_3 \equiv \{(111), (123), (222), (333)\}\}\rangle$

$\mathcal{P}' = \langle X, \mathcal{D}, \mathcal{C}' \rangle, \mathcal{C}' = \{C_1, C_2, C_3' \equiv \{(123), (231), (312)\}\}$

For any tuple $\tau$ on $X(C)$ that does not satisfy $C$ there exists a constraint $C'$ in $\mathcal{C}'$ with $X(C') \subseteq X(C)$ such that $\tau[X(C')] \notin C'$ ($\tau$ local inconsistent).
Hence $\mathcal{P}' \preceq \mathcal{P}$. But also $\mathcal{P} \preceq \mathcal{P}'$.
They are no-good equivalent.

$\rightsquigarrow$ $\preceq$ does not define an order, just a preorder (antisymmetry does not hold.)

# Constraint Propagation



Constraint Propagation transforms a problem $\mathcal{P}$ by tightening $\mathcal{D}$, by tightening constraints from $\mathcal{C}$ or by adding new constraints to $\mathcal{C}$. It does not remove redundant constraints which is a modeling task.

$\mathcal{P}$' is a tightening of $\mathcal{P}$ (and by implication $\mathcal{P}' \preceq \mathcal{P}$) if
$X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{D}_{\mathcal{P}'} \subseteq \mathcal{D}, \quad \forall C \in \mathcal{C}, \exists C' \in \mathcal{C}', X(C') = X(C)$ and $C' \subseteq C$.

Note that in the previous example $\mathcal{P}'$ is not a tightening of $\mathcal{P}$: $C'_3 \not\subseteq$ of any $C \in \mathcal{C}$, neither viceversa. Tightening defines a non-strict order (preorder).

Example

$$
\begin{aligned}
\mathcal{P} =& \langle X = (x_1, x_2, x_3), \mathcal{D} = \{D(x_i) = [1..4], \forall i\}, \\
& \mathcal{C} = \{C_1 \equiv x_1 < x_2, C_2 \equiv x_2 < x_3, \\
& C_3 \equiv \{(111), (123), (222), (333), (234)\}\}\rangle \\
\mathcal{P}' =& \langle X, \mathcal{D}, \mathcal{C}'\rangle, \mathcal{C}' = \{C_1, C_2, C'_3 \equiv \{(123)\}\}\rangle
\end{aligned}
$$

$\mathcal{P}' \preceq \mathcal{P}$: $X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{D}_{\mathcal{P}'} = \mathcal{D}$ and
$C_1 = C'_1, C_2 = C'_2, X(C_3) = X(C'_3), C'_3 \subset C_3$.

# Notation and Terminology



$\mathcal{S}_{\mathcal{P}}$ is the space of all tightening for $\mathcal{P}$

We are interested in the tightenings that preserve the set of solutions ($\mathrm{sol}(\mathcal{P}') = \mathrm{sol}(\mathcal{P})$) whose space is denoted $\mathcal{S}_{\mathcal{P}}^{\mathrm{sol}}$ and among them the smallest ($\mathcal{P}^* \preceq \mathcal{P}''$ for all $\mathcal{P}'' \in \mathcal{S}_{\mathcal{P}}^{\mathrm{sol}}$.)

$\mathcal{P}^* \in \mathcal{S}_{\mathcal{P}}^{\mathrm{sol}}$ is globally consistent if any instantiation $I$ on $Y \subseteq X$ which is locally consistent in $\mathcal{P}^*$ can be extended to a solution of $\mathcal{P}$.

Computing $\mathcal{P}^*$ is exponential in time and space $\rightsquigarrow$ search a close $\mathcal{P}$ in polynomial time and space $\rightsquigarrow$ constraint propagation

- Define a property $\Phi$ that states necessary conditions on instantiations for solutions. $\Phi$ is called local consistency.
- Reduction rules: sufficient conditions to rule out values (or instantiations) that will not be part of a solution (defined through a consistency property $\Phi$)
  Rules iteration: set of reduction rules for each constraint that tighten the problem

# Constraint Propagation



In general, we reach a $\mathcal{P}$' that is $\Phi$ consistent by constraint propagation:

- tighten $\mathcal{D}$
- tighten $\mathcal{C}$, ex: $x_1 + x_2 \leq x_3 \rightsquigarrow x_1 + x_2 = x_3$
- add $C$ to $\mathcal{C}$

Focus on domain-based tightenings

# Domain-based tightenings



The space $\mathcal{S}_{\mathcal{P}}$ of domain-based tightenings of $\mathcal{P}$ is the set of problems $\mathcal{P}' = \langle X', \mathcal{D}', \mathcal{C}' \rangle$ such that $X_{\mathcal{P}'} = X_{\mathcal{P}}, \quad \mathcal{D}_{\mathcal{P}'} \subseteq \mathcal{D}, \quad \mathcal{C}' = \mathcal{C}$

As before the task is:
Finding a tightening $\mathcal{P}^*$ in $\mathcal{S}_{\mathcal{P}}^{\mathrm{sol}} \subseteq \mathcal{S}_{\mathcal{P}}$ (the set that contains all problems that preserve the solutions of $\mathcal{P}$) such that:
forall $x_i \in X_{\mathcal{P}}$, $D_{\mathcal{P}*}(x_i)$ contains only values that belong to a solution itself, i.e., $D_{\mathcal{P}*}(x_i) = \pi_{\{x_i\}}(\mathrm{sol}(\mathcal{P}))$

- Reduction rules:

  $D(x_i) \leftarrow D(x_i) \cap \{v_i | D(x_1) \times D(x_j - 1) \times \{v_i\} \times \ldots D(x_j + 1) \times \ldots D(x_k) \cap C \neq \emptyset\}$

  (the rule is parameterised by a variable $x_i$ and a constraint $C$)
- Rules iteration (for all $i$)

It is clearly NP-hard since it corresponds to solving $\mathcal{P}$ itself.
$\leadsto$ hence polynomial reduction rules to approximate $\mathcal{P}^*$

Apply rules iteration for each constraint. Domain-based reduction rules are also called propagators.

**Example**

$C = (|x_1 - x_2| = k)$
Propagator: $D(x_1) \leftarrow D(x_1) \cap [\min_D(x_2) - k .. \max_D(x_2) + k]$

Rather than defining rules we define $\Phi$: e.g., unary, arc, path, $k$-consistency

# Domain-based local consistency

Domain-based local consistency property $\Phi$ specifies a necessary condition on values to belong to solutions. We restrict to those stable under union.

A domain-based property $\Phi$ is stable under union iff for any $\Phi$-consistent problem $\mathcal{P}_1 = \langle X, \mathcal{D}, \mathcal{C} \rangle$ and $\mathcal{P}_2 = \langle X, \mathcal{D}, \mathcal{C} \rangle$ the problem $\mathcal{P}' = \langle X, \mathcal{D}_1 \cup \mathcal{D}_2, C \rangle$ is $\Phi$-consistent.

**Example**

$\Phi$ for each constraint $C$ and variable $x_i \in X(C)$, at least half of the values in $D(x_i)$ belong to a valid tuple satisfying $C$.

$$\mathcal{P}_1 = \langle X = (x_1, x_2), \mathcal{D} = \{D_1(x_1) = \{1, 2\}, D_1(x_2) = \{2\}\}, C \equiv \{x_1 = x_2\} \rangle$$
$$\mathcal{P}_2 = \langle X = (x_1, x_2), \mathcal{D} = \{D_2(x_1) = \{2, 3\}, D_2(x_2) = \{2\}\}, C \equiv \{x_1 = x_2\} \rangle$$

Both are $\Phi$ consistent but they are not stable under union.

# Domain-based tightenings



Note: Not all $\Phi$-consistent tightenings preserve the solutions
We search for the $\Phi$-closure $\Phi(\mathcal{P})$ (the union of all $\Phi$-consistent $\mathcal{P}' \in \mathcal{S}_{\mathcal{P}}$)

If $\Phi$ is stable under union, then $\Phi(\mathcal{P})$ is the unique domain-based $\Phi$-consistent tightening problem that contains all others.

$\mathrm{sol}(\phi(\mathcal{P})) = \mathrm{sol}(\mathcal{P})$

Example

$$\mathcal{P} = \langle X = (x_1, x_2, x_3, x_4), \mathcal{D} = \{D(x_i) = \{1,2\}, \forall i\}, \\ \mathcal{C} = \{C_1 \equiv x_1 \leq x_2, C_2 \equiv x_2 \leq x_3, C_3 \equiv x_1 \neq x_3\}\rangle$$

$\Phi$ all values for all variables can be extended consistently to a second variable

$$\mathcal{P}' = \langle X = (x_1, x_2, x_3, x_4), \mathcal{D} = \{D(x_1) = 1, D(x_2) = 1, D(x_3) = 2, \forall i\}, \\ \mathcal{C} = \{C_1 \equiv x_1 \leq x_2, C_2 \equiv x_2 \leq x_3, C_3 \equiv x_1 \neq x_3\}\rangle$$

$\mathcal{P}$' is consistent but it does not contain $(1,2,2)$ which is in $\mathrm{sol}(\mathcal{P})$
$\Phi(\mathcal{P}) : \langle X, \mathcal{D}_\Phi, \mathcal{C}\rangle$ with $D_\Phi(x_1) = 1, D_\Phi(x_2) = \{1,2\}, D_\Phi(x_3) = 2$

# Definition



A set is closed under an operation if performance of that operation on members of the set always produces a member of the same set.

A set is said to be closed under a collection of operations if it is closed under each of the operations individually.

# Domain-based tightenings



**Proposition (Fixed Point):** If a domain based consistency property $\Phi$ is stable under union, then for any $\mathcal{P}$, the $\mathcal{P}$' with $\mathcal{D}_{\mathcal{P}'}$ obtained by iteratively removing values that do not satisfy $\Phi$ until no such value exists is the $\Phi$-closure of $\mathcal{P}$.

Contrary to $\mathcal{P}^*$, $\Phi(\mathcal{P})$ can be computed by a greedy algorithm:

**Corollary** If a domain-based consistency property $\Phi$ is polynomial to check, finding $\Phi(\mathcal{P})$ is polynomial as well.

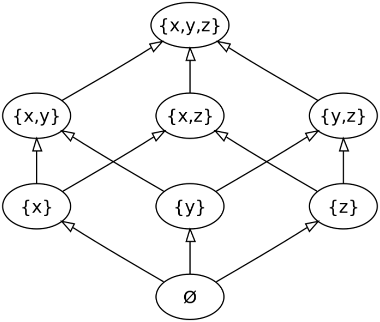
enforcing $\Phi$ consistency $\equiv$ finding closure $\Phi(\mathcal{P})$

# Orders



Domain-based tightenings define a partial order (poset) because isomorphic to inclusion $\subseteq$, which is a partial order

(For $a$, $b$, elements of a poset $P$, if $a \leq b$ or $b \leq a$, then $a$ and $b$ are comparable. Otherwise they are incomparable)



# Orders



Possible to define a partial order also on the local consistency property:

Definition

- $\Phi_1$ is at least as strong as another $\Phi_2$ if for any $\mathcal{P}$: $\Phi_1(\mathcal{P}) \leq \Phi_2(\mathcal{P})$: ie, $X_{\Phi_1(\mathcal{P})} = X_{\Phi_2(\mathcal{P})}, \quad \mathcal{D}_{\Phi_1(\mathcal{P})} \subseteq \mathcal{D}_{\Phi_2(\mathcal{P})}, \quad \mathcal{C}_{\Phi_1(\mathcal{P})} = \mathcal{C}_{\Phi_2(\mathcal{P})}$ (any instantiation $I$ on $Y \subseteq X_{\Phi_2(\mathcal{P})}$ locally inconsistent in $\Phi_2(\mathcal{P})$ is locally inconsistent in $\Phi_1(\mathcal{P})$)
- $\Phi_1$ is stricly stronger than $\Phi_2$ if it is at least as strong as and there exists a $\mathcal{P}$: $\Phi_1(\mathcal{P}) < \Phi_2(\mathcal{P})$.
- $\Phi_1$ and $\Phi_2$ are incomparable if there exists a $\mathcal{P}$' and $\mathcal{P}$'' such that $\Phi_1(\mathcal{P}') < \Phi_2(\mathcal{P}')$ and $\Phi_2(\mathcal{P}'') < \Phi_1(\mathcal{P}'')$.