DM841

Discrete Optimization

Part 2 – Lecture 1

Satisfiability

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Outline

SAT Problem

Satisfiability problem in propositional logic

$$
\begin{aligned}
&(x_5 \vee x_8 \vee \bar{x}_2) \wedge (x_2 \vee \bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_3 \vee \bar{x}_7) \wedge (\bar{x}_5 \vee x_3 \vee x_8) \wedge \\
&(\bar{x}_6 \vee \bar{x}_1 \vee \bar{x}_5) \wedge (x_8 \vee \bar{x}_9 \vee x_3) \wedge (x_2 \vee x_1 \vee x_3) \wedge (\bar{x}_1 \vee x_8 \vee x_4) \wedge \\
&(\bar{x}_9 \vee \bar{x}_6 \vee x_8) \wedge (x_8 \vee x_3 \vee \bar{x}_9) \wedge (x_9 \vee \bar{x}_3 \vee x_8) \wedge (x_6 \vee \bar{x}_9 \vee x_5) \wedge \\
&(x_2 \vee \bar{x}_3 \vee \bar{x}_8) \wedge (x_8 \vee \bar{x}_6 \vee \bar{x}_3) \wedge (x_8 \vee \bar{x}_3 \vee \bar{x}_1) \wedge (\bar{x}_8 \vee x_6 \vee \bar{x}_2) \wedge \\
&(x_7 \vee x_9 \vee \bar{x}_2) \wedge (x_8 \vee \bar{x}_9 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_9 \vee x_4) \wedge (x_8 \vee x_1 \vee \bar{x}_2) \wedge \\
&(x_3 \vee \bar{x}_4 \vee \bar{x}_6) \wedge (\bar{x}_1 \vee \bar{x}_7 \vee x_5) \wedge (\bar{x}_7 \vee x_1 \vee x_6) \wedge (\bar{x}_5 \vee x_4 \vee \bar{x}_6) \wedge \\
&(\bar{x}_4 \vee x_9 \vee \bar{x}_8) \wedge (x_2 \vee x_9 \vee x_1) \wedge (x_5 \vee \bar{x}_7 \vee x_1) \wedge (\bar{x}_7 \vee \bar{x}_9 \vee \bar{x}_6) \wedge \\
&(x_2 \vee x_5 \vee x_4) \wedge (x_8 \vee \bar{x}_4 \vee x_5) \wedge (x_5 \vee x_9 \vee x_3) \wedge (\bar{x}_5 \vee \bar{x}_7 \vee x_9) \wedge \\
&(x_2 \vee \bar{x}_8 \vee x_1) \wedge (\bar{x}_7 \vee x_1 \vee x_5) \wedge (x_1 \vee x_4 \vee x_3) \wedge (x_1 \vee \bar{x}_9 \vee \bar{x}_4) \wedge \\
&(x_3 \vee x_5 \vee x_6) \wedge (\bar{x}_6 \vee x_3 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee x_5 \vee x_9) \wedge (x_7 \vee \bar{x}_5 \vee \bar{x}_2) \wedge \\
&(x_4 \vee x_7 \vee x_3) \wedge (x_4 \vee \bar{x}_9 \vee \bar{x}_7) \wedge (x_5 \vee \bar{x}_1 \vee x_7) \wedge (x_5 \vee \bar{x}_1 \vee x_7) \wedge \\
&(x_6 \vee x_7 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_6 \vee \bar{x}_7) \wedge (x_6 \vee x_2 \vee x_3) \wedge (\bar{x}_8 \vee x_2 \vee x_5)
\end{aligned}
$$

Does there exist a truth assignment satisfying all clauses?

Search for a satisfying assignment (or prove none exists)

SAT Problem

Satisfiability problem in propositional logic

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\begin{aligned}
&(x_5 \vee x_8 \vee \bar{x}_2) \wedge (x_2 \vee \bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_3 \vee \bar{x}_7) \wedge (\bar{x}_5 \vee x_3 \vee x_8) \wedge \\
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&(x_3 \vee \bar{x}_4 \vee \bar{x}_6) \wedge (\bar{x}_1 \vee \bar{x}_7 \vee x_5) \wedge (\bar{x}_7 \vee x_1 \vee x_6) \wedge (\bar{x}_5 \vee x_4 \vee \bar{x}_6) \wedge \\
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&(x_2 \vee \bar{x}_8 \vee x_1) \wedge (\bar{x}_7 \vee x_1 \vee x_5) \wedge (x_1 \vee x_4 \vee x_3) \wedge (x_1 \vee \bar{x}_9 \vee \bar{x}_4) \wedge \\
&(x_3 \vee x_5 \vee x_6) \wedge (\bar{x}_6 \vee x_3 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee x_5 \vee x_9) \wedge (x_7 \vee \bar{x}_5 \vee \bar{x}_2) \wedge \\
&(x_4 \vee x_7 \vee x_3) \wedge (x_4 \vee \bar{x}_9 \vee \bar{x}_7) \wedge (x_5 \vee \bar{x}_1 \vee x_7) \wedge (x_5 \vee \bar{x}_1 \vee x_7) \wedge \\
&(x_6 \vee x_7 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_6 \vee \bar{x}_7) \wedge (x_6 \vee x_2 \vee x_3) \wedge (\bar{x}_8 \vee x_2 \vee x_5)
\end{aligned}
$$

Does there exist a truth assignment satisfying all clauses?

Search for a satisfying assignment (or prove none exists)

Motivation

- From 100 variables, 200 constraints (early 90s) to 1,000,000 vars. and 20,000,000 cls. in 20 years.
- Applications:
  Hardware and Software Verification, Planning, Scheduling, Optimal Control, Protocol Design, Routing, Combinatorial problems, Equivalence Checking, etc.
- SAT used to solve many other problems!

SAT Problem

Satisfiability problem in propositional logic

Definitions:

- Formula in propositional logic: well-formed string that may contain
  - propositional variables $x_1, x_2, \ldots, x_n$;
  - truth values $\top$ (‘true’), $\bot$ (‘false’);
  - operators $\neg$ (‘not’), $\wedge$ (‘and’), $\vee$ (‘or’);
  - parentheses (for operator nesting).
- Model (or satisfying assignment) of a formula F: Assignment of truth values to the variables in F under which F becomes true (under the usual interpretation of the logical operators)
- Formula F is satisfiable iff there exists at least one model of F, unsatisfiable otherwise.

SAT Problem (decision problem, search variant):

- **Given:** Formula F in propositional logic
- **Task:** Find an assignment of truth values to variables in F that renders F true, or decide that no such assignment exists.

SAT: A simple example

- **Given:** Formula $F := (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$
- **Task:** Find an assignment of truth values to variables x_1, x_2 that renders F true, or decide that no such assignment exists.

Definitions:

- A formula is in conjunctive normal form (CNF) iff it is of the form

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{k_i} l_{ij} = (l_{11} \vee \ldots \vee l_{1k_1}) \wedge \ldots \wedge (l_{m1} \vee \ldots \vee l_{mk_m})$$

  where each literal l_{ij} is a propositional variable or its negation. The disjunctions $c_i = (l_{i1} \vee \ldots \vee l_{ik_i})$ are called clauses.

- A formula is in k-CNF iff it is in CNF and all clauses contain exactly k literals (*i.e.*, for all i, $k_i = k$).

- In many cases, the restriction of SAT to CNF formulae is considered.
- For every propositional formula, there is an equivalent formula in 3-CNF.

Example:

$$\begin{aligned} F := \quad & \wedge (\neg x_2 \vee x_1) \\ & \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \\ & \wedge (x_1 \vee x_2) \\ & \wedge (\neg x_4 \vee x_3) \\ & \wedge (\neg x_5 \vee x_3) \end{aligned}$$

- F is in CNF.
- Is F satisfiable?
  Yes, *e.g.*, $x_1 := x_2 := \top$, $x_3 := x_4 := x_5 := \bot$ is a model of F.

From Propositional Logic to SAT

Propositional logic: operators: $\neg P, P \wedge Q, P \vee Q, P \implies Q, P \Leftrightarrow Q$

To conjunctive normal form:

- replace $\alpha \Leftrightarrow$ with $(\alpha \implies \beta) \wedge (\beta \implies \alpha)$
- replace $\alpha \implies \beta$ with $\neg\alpha \vee \beta$
- $\neg$ must appear only in literals, hence move $\neg$ inwards
- distributive law for $\vee$ over $\wedge$:

$$(\alpha \vee (\beta \wedge \gamma) \text{ infers that } (\alpha \vee \beta) \wedge (\alpha \vee \gamma))$$

Special Cases

Not all instances are hard:

- Definite clauses: exactly one literal in the clause is positive. Eg:

$$\neg\beta \vee \neg\gamma \vee \alpha$$

- Horn clauses: at most one literal is positive.
  - Easy interpretation: $\alpha \wedge \beta \implies \gamma$ infers that $\neg\alpha \vee \neg\beta \vee \gamma$
  - inference is easy by forward checking, linear time

Max SAT

Definition ((Maximum) K-Satisfiability (SAT))

Input: A set U of variables, a collection C of disjunctive clauses of at most k literals, where a literal is a variable or a negated variable in U. k is a constant, $k > 2$.
Task: A truth assignment for U or a truth assignment that maximizes the number of clauses satisfied.

MAX-SAT (optimization problem)

Which is the maximal number of clauses satisfiable in a propositional logic formula F?

Outline

Mathematical Programming Models

- How to model an optimization problem
  - choose some decision variables
    they typically encode the result we are interested into
  - express the problem constraints in terms of these variables
    they specify what the solutions to the problem are
  - express the objective function
    the objective function specifies the quality of each solution
- The result is an optimization model
  - It is a declarative formulation
    specify the "what", not the "how"
  - There may be many ways to model an optimization problem

IP model

Standard IP formulation: Let x_l be a 0–1 variable equal to 1 whenever the literal l takes value true and 0 otherwise.
Let c^+ be the set of literals in clause $c \in C$ that appear as positive and c^- the set of variables that appear as negated.

$$\begin{aligned}
\min \quad & 1 \\
\text{s.t.} \quad & \sum_{l \in c^+} x_l + \sum_{l \in c^-} (1 - x_l) = 1, && \forall c \in C, \\
& x_l \in \{0, 1\}, && \forall l \in L
\end{aligned}$$

Outline

Gecode Model

From Gecode examples:

```
BoolVarArray x = BoolVarArray(*this, nvariables, 0, 1);

for (int c=0; c < nclauses; c++) {
  // Create positive BoolVarArgs
  BoolVarArgs positives(clauses[c].pos.size());
  for (int i=clauses[c].pos.size(); i--;)
    positives[i] = x[clauses[c].pos[i]];

  BoolVarArgs negatives(clauses[c].neg.size());
  for (int i=clauses[c].neg.size(); i--;)
    negatives[i] = x[clauses[c].neg[i]];

  // Post propagators
  clause(*this, BOT_OR, positives, negatives, 1);
}

branch(*this, x, INT_VAR_NONE(), INT_VAL_MIN());
```

Outline

DPLL algorithm

Davis, Putam, Logenmann & Loveland (DPLL) algorithm is a recursive depth-first enumeration of possible models with the following elements:

1. Early termination:
   a clause is true if any of its literals are true
   a sentence is false if any of its clauses are false, which occurs when all its literals are false

2. Pure literal heuristic:
   pure literal is one that appears with same sign everywhere.
   it can be assigned so that it makes the clauses true. Clauses already true can be ignored.

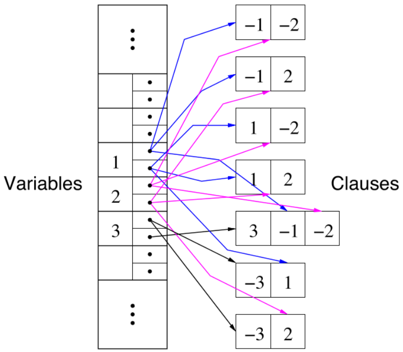
3. Unit clause heuristic
   consider first unit clauses with just one literal or all literal but one already assigned. Generates cascade effect (forward chaining)

DPLL algorithm

```
Function DPLL($C, L, M$):
    Data: $C$ set of clauses; $L$ set of literals; $M$ model;
    Result: $true$ or $false$
    if every clause in $C$ is true in M then return $true$;
    if some clause in $C$ is false in M then return $false$;
    $(l, val) \leftarrow$FindPureLiteral($L, C, M$);
    if $l$ is non-null then return DPLL($C, L \setminus l, M \cup \{l = val\}$);
    $(l, val) \leftarrow$FindUnitClause($L, M$);
    if $l$ is non-null then return DPLL($C, L \setminus l, M \cup \{l = val\}$);
    $l \leftarrow$First($L$); $R \leftarrow$Rest($L$);
    return DPLL($C, R, M \cup \{l = true\}$) or
            DPLL($C, R, M \cup \{l = false\}$)
```

Speedups

- Component analysis to find separable problems
- Intelligent backtracking
- Random restarts
- Clever indexing (data structures)
- Variable value ordering



Variable selection heuristics

- Degree
- Based on the occurrences in the (reduced) formula
  - Maximal Occurrence in clauses of Minimal Size (MOMS, Jeroslow-Wang)
- Variable State Independent Decaying Sum (VSIDS)
  - original idea (zChaff): for each conflict, increase the score of involved variables by 1, half all scores each 256 conflicts [MoskewiczMZZM2001]
  - improvement (MiniSAT): for each conflict, increase the score of involved variables by δ and increase $\delta := 1.05\delta$ [EenSörensson2003]

Value selection heuristics

- Based on the occurrences in the (reduced) formula
    - examples: Jeroslow-Wang, Maximal Occurrence in clauses of Minimal Size (MOMS), look-aheads

Summary