FF505

Computational Science

Systems of Linear Equations

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Outline

1. Solving Linear Systems

Systems of Linear Equations

How many solutions have these linear systems? Find it out using the graphical approach.

$$\begin{aligned} 6x - 10y &= 2 \\ 3x - 4y &= 5 \end{aligned}$$

```
% plot functions in implicit form
ezplot
```

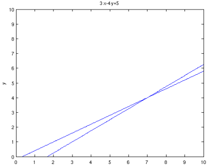
$$\begin{aligned} 3x - 4y &= 5 \\ 6x - 8y &= 10 \end{aligned}$$

$$\begin{aligned} 3x - 4y &= 5 \\ 6x - 8y &= 3 \end{aligned}$$

Systems of Linear Equations

$$\begin{aligned} 6x - 10y &= 2 \\ 3x - \ \ 4y &= 5 \end{aligned}$$

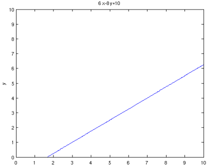
```
% plot functions in implicit form
ezplot('6*x-10*y=2',[0 10 0 10]),
hold,
ezplot('3*x-4*y=5',[0 10 0 10])
```



has one single solution

$$\begin{aligned} 3x - 4y &= \ \ 5 \\ 6x - 8y &= 10 \end{aligned}$$

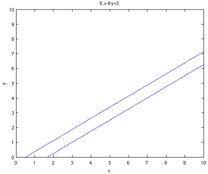
```
ezplot('3*x-4*y=5',[0 10 0 10]),
hold,
ezplot('6*x-8*y=10',[0 10 0 10])
```



has infinite solutions

$$\begin{aligned} 3x - 4y &= 5 \\ 6x - 8y &= 3 \end{aligned}$$

```
ezplot('3*x-4*y=5',[0 10 0 10]),
hold,
ezplot('6*x-8*y=3',[0 10 0 10])
```



has no solution

Matrix Form

The linear system:

$$\begin{aligned} 2x_1 + \quad 9x_2 &= \quad 5 \\ 3x_1 - \quad 4x_2 &= \quad 7 \end{aligned}$$

can be expressed in vector-matrix form as:

$$\begin{bmatrix} 2 & 9 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

In general, a set of m equations in n unknowns can be expressed in the form $A\mathbf{x} = \mathbf{b}$, where A is $m \times n$, $\mathbf{x}$ is $n \times 1$ and $\mathbf{b}$ is $m \times 1$.

The inverse of A is denoted A^{-1} and has property that

$$A^{-1}A = AA^{-1} = I$$

Hence the solution to our system is:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Inverse and Determinant

Compute the inverse and the determinant of this matrix in Matlab:

```
>> A=[3 -4; 6 -8];
```

Has the system solutions?
What about the system in the previous slide? What are its solutions?

A matrix is singular if $\det(A) = |A| = 0$

Inverse of a square matrix A is defined only if A is nonsingular.

If A is singular, the system has no solution

```
>> A=[3 -4; 6 -8];
>> det(A)
ans =
     0
>> inv(A)
Warning: Matrix is singular to working precision.
ans =
   Inf  Inf
   Inf  Inf
```

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the matrix inverse is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the matrix inverse is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$

Calculating the inverse

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$$

$\mathrm{adj(A)}$ is the adjugate matrix of A:

1. Calculate the (i,j) minor of A, denoted M_{ij}, as the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A.
2. Calculate the cofactor matrix of A, as the $n\times n$ matrix C whose (i,j) entry is the (i,j) cofactor of A

$$C_{ij} = (-1)^{i+j}M_{ij}$$

3. set $\mathrm{adj(A)}_{ij} = C_{ji}$

Left Division Method

- $\mathbf{x} = A^{-1}\mathbf{b}$ rarely applied in practice because calculation is likely to introduce numerical inaccuracy

- The inverse is calculated by LU decomposition, the matrix form of Gaussian elimination.

```
% left division method
x = A\b
```

$$A = LU$$
$$PA = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- for a matrix A, $n \times n$, $\det(A) \neq 0 \Leftrightarrow$ rank of A is n
  - for a system $A\mathbf{x} = \mathbf{b}$ with m equations and n unknowns a solution exists iff $rank(A) = rank([A\mathbf{b}]) = r$
    - if $r = n \rightsquigarrow$ unique
    - if $r < n \rightsquigarrow$ infinite sol.
  - for a homogeneous system $A\mathbf{x} = \mathbf{0}$ it is always $rank(A) = rank([A\mathbf{b}])$ and there is a nonzero solution iff $rank(A) < n$

• `A\b` works for square and nonsquare matrices. If nonsquare ($m < n$) then the system is underdetermined (infinite solutions). `A\b` returns one variable to zero
• `A\b` does not work when $\det(A) = 0$.

```
>> A=[2, -4,5;-4,-2,3;2,6,-8];
>> b=[-4;4;0];
>> rank(A)
ans =
    2
>> rank([A,b])
ans =
    2
>> x=A\b
Warning: Matrix is singular to working
     precision.
x =
   NaN
   NaN
   NaN
```

However since

$$rank(A) = rank([A\mathbf{b}])$$

an infinite number of solutions exist (underdetermined system).
`x=pinv(A)b` solves with pseudoinverse and `rref([A,b])` finds the reduced row echelon form

Overdetermined Systems

An overdetermined system is a set of equations that has more independent equations than unknowns ($m > n$).

For such a system the matrix inverse method will not work because the A matrix is not square.

However, some overdetermined systems have exact solutions, and they can be obtained with the left division method `x = A \ b`

If a solution does not exist, the left-division answer is the least squares solution. We need to check the ranks of A and $[A\mathbf{b}]$ to know whether the answer is the exact solution.

Flowchart for Linear System Solver