DM559/DM545

Linear and Integer Programming

Introduction to Linear Programming
Notation and Modeling

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Who is here?

DM559 (7.5 ECTS)
66 officially registered
53 handed in 0.1

- Computer Science
  (2nd year, 4th semester)

Prerequisites

- Programming

DM545 (5 ECTS)
28 officially registered

- Math-economy
  (3rd year ?)
- Applied Mathematics
  (2nd year ?)
- Others?

Prerequisites

- Programming
- Linear Algebra (MM505)

Outline

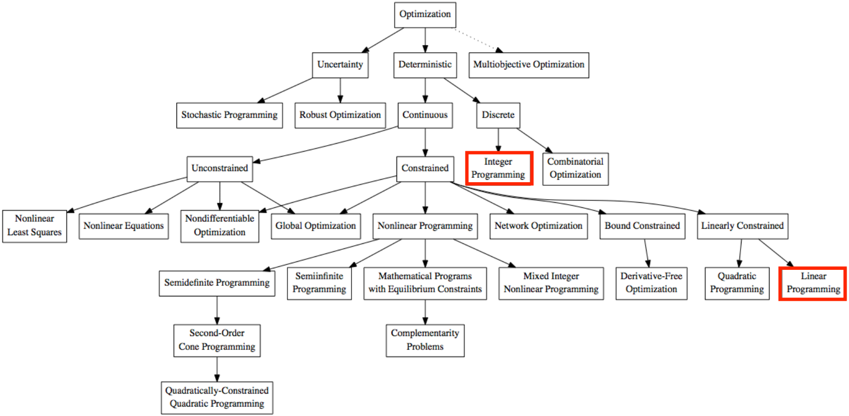
Aims of the course

Learn about mathematical optimization:

- linear programming (continuous optimization)
- integer programming (discrete optimization)

$\rightsquigarrow$ You will apply the tools learned to solve real life problems using computer software

Optimization Taxonomy



Contents of the Course

(see Syllabus)

Linear Programming

1 Introduction - Linear Programming, Notation & Modeling
2 Linear Programming, Simplex Method
3 Exception Handling
4 Duality Theory
5 Sensitivity
6 Revised Simplex Method

Integer Linear Programming

7 Modeling Examples, Good Formulations, Relaxations
8 Well Solved Problems
9 Network Optimization Models (Max Flow, Min cost flow, Matching)
10 Cutting Planes & Branch and Bound
11 More on Modeling

Practical Information

Teacher: Marco Chiarandini (www.imada.sdu.dk/~marco/)
Instructor: Jens Østergaard
Sections (hold): H1, H2, M1

Alternative views of the schedule:

- mitsdu.sdu.dk, SDU Mobile
- Official course description (læserplanen)
- http://www.imada.sdu.dk/~marco/DM545

Schedule:

- Introductory classes: $\sim$ 28 hours ($\sim$ 14 classes)
- Training classes: $\sim$ 24 hours ($\sim$ 12 classes)
  - Exercises: 20 hours
  - Laboratory: 4 hours (2 classes, week 13 and 17)

Communication Means

- BlackBoard (BB) $\Leftrightarrow$ Main Web Page (WP) (link http://www.imada.sdu.dk/~marco/DM545)
- Announcements in BlackBoard
- Write to Marco (marco@imada.sdu.dk) and to instructor
- Ask peers
- You are welcome to visit me in my office in working hours (8-16)

$\rightsquigarrow$ It is good to ask questions!!

$\rightsquigarrow$ Let me know if you think we should do things differently!

Sources

Linear Programming:

LN Lecture Notes (continously updated)

MG J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

Integer Programming:

LN Lecture Notes (continously updated)

Wo L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

Other books and articles:

HL Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

... see webpage

Course Material

Main Web Page (WP) is the main reference for list of contents (ie[1], syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- resources for programming tasks

[1] ie = id est, eg = exempli gratia, wrt = with respect to

Assessment

Accomplishment of the following is required for 5 ECTS:

- Three obligatory Assignments, pass/fail, evaluation by teacher
  - modeling + describing + programming in Python with Gurobi
  - (language: Danish and/or English)
  - individual

  Deadlines:
    - March 24
    - April 26
    - May 25

- 4 hour written exam, 7-grade scale, external censor
  - similar to exercises in class and past exams
  - on June 13

Training Sessions

- Prepare the starred exercises in advance to get out the most
- Try the others after the session
- Best if carried out in small groups
- Exercises are examples of exam questions (but not only!)

Concepts from Linear Algebra

Linear Algebra:
manipulation of matrices and vectors with some theoretical background

Linear Algebra
- Matrices and vectors - Matrix algebra
- Inner (dot) product
- Geometric insight
- Systems of Linear Equations - Row echelon form, Gaussian elimination
- Matrix inversion and determinants
- Rank and linear dependency

DM545 has an obligatory assignment on this.

Coding

- gives you the ability to create new and useful artifacts with just your mind and your fingers,
- allows you to have more control of your world as more and more of it becomes digital,
- is just fun.

It can also help you understand math.
Being able to turn procedural ideas into code and run the code on concrete examples give you a great advantage in developing and reinforcing your understanding of mathematical concepts.
Beside:

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand

You can learn by doing interacting with Python.

from Coding the Matrix by Philip Klein

- Python 2.7 (`import from __future__`) + Gurobi (100 000 Dkk)
- ipython, jupyter (= interactive python)?

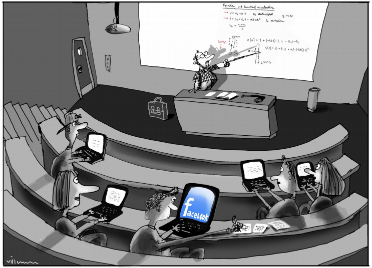
Attendance

- Come to classes and exercise sessions.
  You learn how to reason about topics from this field

- Use computers in class only for course related purposes

- Note that research shows: taking notes by hand yields better long-term comprehension
  http://www.psychologicalscience.org/index.php/news/releases/take-notes-by-hand-for-better-long-term-comprehension.html

- However: the exam is digital!



Jørn Villumsen, Politiken

Outline

What is Operations Research?

Operations Research (aka, Management Science, Analytics):
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system, usually under conditions requiring the allocation of scarce resources, by means of **mathematics** and **computer science**.

Quantitative methods for planning and analysis.

It encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency:

- simulation,
- mathematical optimization,
- queueing theory and other stochastic-process models,
- Markov decision processes
- econometric methods,
- data envelopment analysis,
- neural networks,
- expert systems

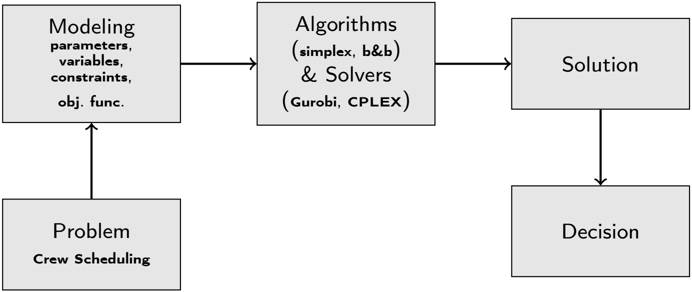
Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
    - Fewest number of people
    - Shortest route
- Not all plans are feasible - there are constraining rules
    - Limited amount of available resources
- It can be extremely difficult to figure out what to do

OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Central Idea

Build a mathematical model describing exactly what one wants, and what the "rules of the game" are. However, **what is a mathematical model and how?**

Mathematical Modeling

- Find out exactly what the decision maker needs to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Decision Variables of suitable type (continuous, integer, binary) corresponding to the needs
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

Outline

Resource Allocation

In manufacturing industry, factory planning: find the best product mix.

Example

A factory makes two products standard and deluxe.

A unit of standard gives a profit of 6k Dkk.
A unit of deluxe gives a profit of 8k Dkk.

The grinding and polishing times in terms of hours per week for a unit of each type of product are given below:

	Standard	Deluxe
(Machine 1) Grinding	5	10
(Machine 2) Polishing	4	4

Grinding capacity: 60 hours per week
Polishing capacity: 40 hours per week
Q: How much of each product, standard and deluxe, should we produce to maximize the profit?

Mathematical Model

Decision Variables

$x_1 \geq 0$ units of product standard
$x_2 \geq 0$ units of product deluxe

Object Function

$\max\ 6x_1 + 8x_2$ maximize profit

Constraints

$$\begin{array}{rcrcl} 5x_1 & + & 10x_2 & \leq & 60 \quad \text{Grinding capacity} \\ 4x_1 & + & 4x_2 & \leq & 40 \quad \text{Polishing capacity} \end{array}$$

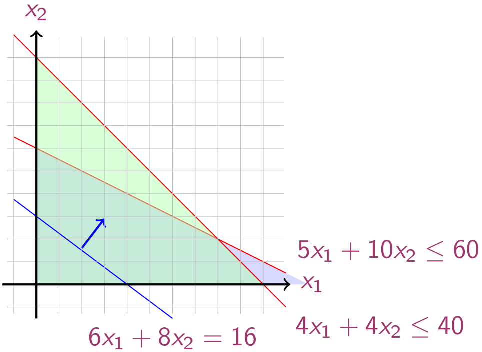
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{array}{rrrrrr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1 & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$j = 1, 2, \ldots, n$	products
$i = 1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}$	profit per unit of product j
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1 x_1 & + & c_2 x_2 & + & c_3 x_3 & + & \ldots & + & c_n x_n & = & z \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + & a_{13} x_3 & + & \ldots & + & a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + & a_{23} x_3 & + & \ldots & + & a_{2n} x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + & a_{m3} x_3 & + & \ldots & + & a_{mn} x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{rrl}
\max & \sum_{j=1}^{n} c_jx_j & \\
 & \sum_{j=1}^{n} a_{ij}x_j & \leq b_i, \quad i = 1, \dots, m \\
 & x_j & \geq 0, \quad j = 1, \dots, n
\end{array}
$$

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & \mathbf{c}^T\mathbf{x} \\
 & A\mathbf{x} & \leq & \mathbf{b} \\
 & \mathbf{x} & \geq & 0
\end{array}
$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m \\ & x_j \ge 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \le 60 \\ & 4x_1 + 4x_2 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned}$$

$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \ge 0$$

Outline

Duality

Resource Valuation problem: Determine the value of the raw materials on hand such that: The company must be willing to sell the raw materials should an outside firm offer to buy them at a price consistent with the market

z_i	value of a unit of raw material i
$\sum_{i=1}^m b_i z_i$	opportunity cost (cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal is to minimize the lost opportunity cost (ie, the cost for the outside company)

$$\min \sum_{i=1}^{m} b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} z_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(2) and (3) otherwise contradicting market

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\min \sum_{i=1}^{m} y_i b_i + \sum_i \rho_i b_i$$
$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1 \dots n$$
$$y_i \geq 0, \quad i = 1 \dots m$$

Dual Problem

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \dots, m$$
$$x_j \geq 0, \quad j = 1, \dots, n$$

Primal Problem