DM545

Linear and Integer Programming

Linear Programming

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Outline

1. Introduction
   Diet Problem

2. Solving LP Problems
   Fourier-Motzkin method

3. Preliminaries
   Fundamental Theorem of LP
   Gaussian Elimination

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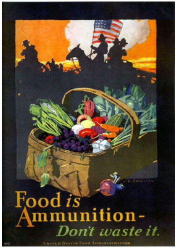
The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
eat enough but not too much of Vitamin A
eat enough but not too much of Sodium
eat enough but not too much of Calories
...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread, and
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5,000 and 50,000)

Food	Cost per serving	Vitamin A	Calories
Corn	\$0.18	107	72
2% Milk	\$0.23	500	121
Wheat Bread	\$0.05	0	65

The Mathematical Model

Parameters (given data)

F = set of foods
N = set of nutrients

a_{ij} = amount of nutrient i in food j, $\forall i \in N$, $\forall j \in F$
c_j = cost per serving of food $j, \forall j \in F$
$F_{min,j}$ = minimum number of required servings of food $j, \forall j \in F$
$F_{max,j}$ = maximum allowable number of servings of food $j, \forall j \in F$
$N_{min,i}$ = minimum required level of nutrient $i, \forall i \in N$
$N_{max,i}$ = maximum allowable level of nutrient $i, \forall i \in N$

Decision Variables

x_j = number of servings of food i to purchase/consume, $\forall j \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{j \in F} c_j x_j$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{j \in F} a_{ij} x_j \geq N_{min,i}, \forall i \in N$$

Constraint Set 2: For each nutrient $i \in N$, do not exceed the maximum allowable level.

$$\sum_{j \in F} a_{ij} x_j \leq N_{max,i}, \forall i \in N$$

Constraint Set 3: For each food $j \in F$, select at least the minimum required number of servings

$$x_j \geq F_{min,j}, \forall j \in F$$

Constraint Set 4: For each food $j \in F$, do not exceed the maximum allowable number of servings.

$$x_j \leq F_{max,j}, \forall j \in F$$

The Mathematical Model

system of equalities and inequalities

$$\min \sum_{j \in F} c_j x_j$$

$$\sum_{j \in F} a_{ij} x_j \geq N_{min,i}, \qquad \forall i \in N$$

$$\sum_{j \in F} a_{ij} x_j \leq N_{max,i}, \qquad \forall i \in N$$

$$x_j \geq F_{min,j}, \qquad \forall j \in F$$

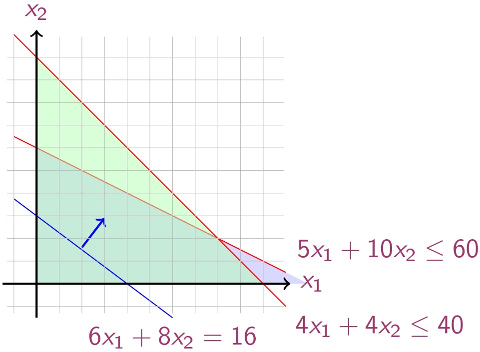
$$x_j \leq F_{max,j}, \qquad \forall j \in F$$

Mathematical Model

Graphical Representation:

Machines/Materials A and B
Products 1 and 2

$$\begin{aligned} \max\ 6x_1 + \ 8x_2 & \\ 5x_1 + 10x_2 &\leq 60 \\ 4x_1 + \ 4x_2 &\leq 40 \\ x_1 &\geq \ 0 \\ x_2 &\geq \ 0 \end{aligned}$$



In Matrix Form

$$
\begin{array}{rl}
\max & c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = z \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \leq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{array}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{rl}
\max & z = \mathbf{c}^T\mathbf{x} \\
& A\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{array}
$$

Linear Programming

Abstract mathematical model:
Parameters, Decision Variables, Objective, Constraints
(+ Domains & Quantifiers)

The Syntax of a Linear Programming Problem

objective func.	$\max / \min \mathbf{c}^T \cdot \mathbf{x}$	$\mathbf{c} \in \mathbb{R}^n$
constraints	s.t. $A \cdot \mathbf{x} \gtreqless \mathbf{b}$	$A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$
	$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \in \mathbb{R}^n, \mathbf{0} \in \mathbb{R}^n$

Essential features: continuity, linearity (proportionality and additivity), certainty of parameters

- Any vector $\mathbf{x} \in \mathbb{R}^n$ satisfying all constraints is a feasible solution.
- Each $\mathbf{x}^* \in \mathbb{R}^n$ that gives the best possible value for $\mathbf{c}^T\mathbf{x}$ among all feasible $\mathbf{x}$ is an optimal solution or optimum
- The value $\mathbf{c}^T\mathbf{x}^*$ is the optimum value

- The linear programming model consisted of 9 equations in 77 variables
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution
- The original instance:
  http://www.gams.com/modlib/libhtml/diet.htm

AMPL Model

```
# diet.mod
set NUTR;
set FOOD;

param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max { j in FOOD} >= f_min[j];
param n_min { NUTR } >= 0;
param n_max {i in NUTR } >= n_min[i];
param amt {NUTR,FOOD} >= 0;

var Buy { j in FOOD} >= f_min[j], <= f_max[j]

minimize total_cost: sum { j in FOOD } cost [j] * Buy[j];
subject to diet { i in NUTR }:
        n_min[i] <= sum {j in FOOD} amt[i,j] * Buy[j] <= n_max[i];
```

AMPL Model

```
# diet.dat
data;

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH
      MTL SPG TUR;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
   A 700 10000
   C 700 10000
   B1 700 10000
   B2 700 10000 ;
# %
```

```
param amt (tr):
           A C B1 B2 :=
   BEEF 60 20 10 15
   CHK 8 0 20 20
   FISH 8 10 15 10
   HAM 40 40 35 10
   MCH 15 35 15 15
   MTL 70 30 15 15
   SPG 25 50 25 15
   TUR 60 20 15 10 ;
```

Python Script
Data


```
from gurobipy import *

categories, minNutrition, maxNutrition =
        multidict({
  'calories': [1800, 2200],
  'protein': [91, GRB.INFINITY],
  'fat': [0, 65],
  'sodium': [0, 1779] })

foods, cost = multidict({
  'hamburger': 2.49,
  'chicken': 2.89,
  'hot dog': 1.50,
  'fries': 1.89,
  'macaroni': 2.09,
  'pizza': 1.99,
  'salad': 2.49,
  'milk': 0.89,
  'ice cream': 1.59 })
```

```
# Nutrition values for the foods
nutritionValues = {
  ('hamburger', 'calories'): 410,
  ('hamburger', 'protein'): 24,
  ('hamburger', 'fat'): 26,
  ('hamburger', 'sodium'): 730,
  ('chicken', 'calories'): 420,
  ('chicken', 'protein'): 32,
  ('chicken', 'fat'): 10,
  ('chicken', 'sodium'): 1190,
  ('hot dog', 'calories'): 560,
  ('hot dog', 'protein'): 20,
  ('hot dog', 'fat'): 32,
  ('hot dog', 'sodium'): 1800,
  ('fries', 'calories'): 380,
  ('fries', 'protein'): 4,
  ('fries', 'fat'): 19,
  ('fries', 'sodium'): 270,
  ('macaroni', 'calories'): 320,
  ('macaroni', 'protein'): 12,
  ('macaroni', 'fat'): 10,
  ('macaroni', 'sodium'): 930,
  ('pizza', 'calories'): 320,
  ('pizza', 'protein'): 15,
  ('pizza', 'fat'): 12,
  ('pizza', 'sodium'): 820,
  ('salad', 'calories'): 320,
  ('salad', 'protein'): 31,
```

```
# Model diet.py
m = Model("diet")

# Create decision variables for the foods to buy
buy = {}
for f in foods:
    buy[f] = m.addVar(obj=cost[f], name=f)

# The objective is to minimize the costs
m.modelSense = GRB.MINIMIZE

# Update model to integrate new variables
m.update()

# Nutrition constraints
for c in categories:
    m.addConstr(
      quicksum(nutritionValues[f,c] * buy[f] for f in foods) <= maxNutrition[c], name=c+'max')
    m.addConstr(
      quicksum(nutritionValues[f,c] * buy[f] for f in foods) >= minNutrition[c], name=c+'min')

# Solve
m.optimize()
```

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History of Linear Programming (LP)
System of linear equations

$\rightsquigarrow$ It is impossible to find out who knew what when first.
Just two "references":

- Egyptians and Babylonians considered about 2000 B.C. the solution of special linear equations. But, of course, they described examples and did not describe the methods in "today's style".

- What we call "Gaussian elimination"today has been explicitly described in Chinese "Nine Books of Arithmetic"which is a compendium written in the period 2010 B.C. to A.D. 9, but the methods were probably known long before that.

- Gauss, by the way, never described "Gaussian elimination". He just used it and stated that the linear equations he used can be solved "per eliminationem vulgarem"

History of Linear Programming (LP)

- Origins date back to Newton, Leibnitz, Lagrange, etc.
- In 1827, Fourier described a variable elimination method for systems of linear inequalities, today often called Fourier-Moutzkin elimination (Motzkin, 1937). It can be turned into an LP solver but inefficient.
- In 1932, Leontief (1905-1999) Input-Output model to represent interdependencies between branches of a national economy (1976 Nobel prize)
- In 1939, Kantorovich (1912-1986): Foundations of linear programming (Nobel prize in economics with Koopmans on LP, 1975) on Optimal use of scarce resources: foundation and economic interpretation of LP
- The math subfield of Linear Programming was created by George Dantzig, John von Neumann (Princeton), and Leonid Kantorovich in the 1940s.
- In 1947, Dantzig (1914-2005) invented the (primal) simplex algorithm working for the US Air Force at the Pentagon. (program=plan)

History of LP (cntd)

- In 1954, Lemke: dual simplex algorithm, In 1954, Dantzig and Orchard Hays: revised simplex algorithm
- In 1970, Victor Klee and George Minty created an example that showed that the classical simplex algorithm has exponential worst-case behavior.
- In 1979, L. Khachain found a new efficient algorithm for linear programming. It was terribly slow. (Ellipsoid method)
- In 1984, Karmarkar discovered yet another new efficient algorithm for linear programming. It proved to be a strong competitor for the simplex method. (Interior point method)

History of Optimization

- In 1951, Nonlinear Programming began with the Karush-Kuhn-Tucker Conditions
- In 1952, Commercial Applications and Software began
- In 1950s, Network Flow Theory began with the work of Ford and Fulkerson.
- In 1955, Stochastic Programming began
- In 1958, Integer Programming began by R. E. Gomory.
- In 1962, Complementary Pivot Theory

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Fourier Motzkin elimination method

Has $Ax \leq b$ a solution? (Assumption: $A \in \mathbb{Q}^{m \times n}, \mathbf{b} \in \mathbb{Q}^n$)
Idea:

1. transform the system into another by eliminating some variables such that the two systems have the same solutions over the remaining variables.
2. reduce to a system of constant inequalities that can be easily decided

Let x_r be the variable to eliminate
Let $M = \{1 \ldots m\}$ index the constraints
For a variable j let partition the rows of the matrix in

$$N = \{i \in M \mid a_{ij} < 0\}$$
$$Z = \{i \in M \mid a_{ij} = 0\}$$
$$P = \{i \in M \mid a_{ij} > 0\}$$

$$\begin{cases} x_r \geq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, & a_{ir} < 0 \\ x_r \leq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, & a_{ir} > 0 \\ \text{all other constraints} & i \in Z \end{cases} \qquad \begin{cases} x_r \geq A_i(x_1, \ldots, x_{r-1}), & i \in N \\ x_r \leq B_i(x_1, \ldots, x_{r-1}), & i \in P \\ \text{all other constraints} & i \in Z \end{cases}$$

Hence the original system is equivalent to

$$\begin{cases} \max\{A_i(x_1, \ldots, x_{r-1}), i \in N\} \leq x_r \leq \min\{B_i(x_1, \ldots, x_{r-1}), i \in P\} \\ \text{all other constraints} \quad i \in Z \end{cases}$$

which is equivalent to

$$\begin{cases} A_i(x_1, \ldots, x_{r-1}) \leq B_j(x_1, \ldots, x_{r-1}) & i \in N, j \in P \\ \text{all other constraints} & i \in Z \end{cases}$$

we eliminated x_r but:

$$\begin{cases} |N| \cdot |P| \text{ inequalities} \\ |Z| \text{ inequalities} \end{cases}$$

after d iterations if $|P| = |N| = m/2$ exponential growth: $(1/2^d)(m/2)^{2^d}$

Example

$$\begin{array}{rrl} -7x_1 & +\ 6x_2 & \leq 25 \\ x_1 & -\ 5x_2 & \leq 1 \\ x_1 & & \leq 7 \\ -x_1 & +\ 2x_2 & \leq 12 \\ -x_1 & -\ 3x_2 & \leq 1 \\ 2x_1 & -\ x_2 & \leq 10 \end{array}$$

x_2 variable to eliminate
$N = \{2, 5, 6\}, Z = \{3\}, P = \{1, 4\}$
$|Z \cup (N \times P)| = 7$ constraints

By adding one variable and one inequality, Fourier-Motzkin elimination can be turned into an LP solver.

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Definitions

- $\mathbb{R}$: set of real numbers
  $\mathbb{N} = \{1, 2, 3, 4, ...\}$: set of natural numbers (positive integers)
  $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$: set of all integers
  $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$: set of rational numbers

- column vector and matrices
  scalar product: $\mathbf{y}^T\mathbf{x} = \sum_{i=1}^n y_i x_i$

- linear combination

$$\begin{aligned} \mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_k &\in \mathbb{R}^n \\ \boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_k]^T &\in \mathbb{R}^k \end{aligned} \qquad \mathbf{x} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k = \sum_{i=1}^k \lambda_i \mathbf{v}_i$$

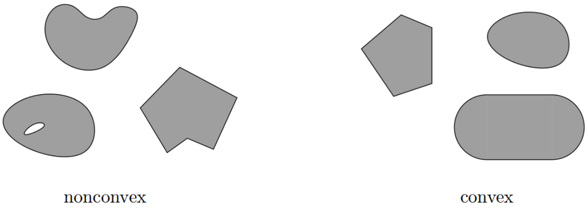
moreover:

$\boldsymbol{\lambda} \geq \mathbf{0}$	conic combination	
$\boldsymbol{\lambda}^T\mathbf{1} = 1$	affine combination	$\left(\sum_{i=1}^k \lambda_i = 1\right)$
$\boldsymbol{\lambda} \geq \mathbf{0}$ and $\boldsymbol{\lambda}^T\mathbf{1} = 1$	convex combination	

Definitions

- set S is linear (affine) independent if no element of it can be expressed as linear combination of the others
  Eg: $S \subseteq \mathbb{R}^n \implies$ max n lin. indep. (max $n+1$ aff. indep.)

- convex set: if $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq \lambda \leq 1$ then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$

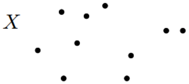


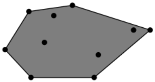
- convex function if its epigraph $\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is a convex set
  or $f : X \to \mathbb{R}$, if $\forall x, y \in X, \lambda \in [0, 1]$ it holds that
  $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

Definitions

- For a set of points $S \subseteq \mathbb{R}^n$

$$\begin{aligned} \text{lin}(S) &\quad \text{linear hull (span)} \\ \text{cone}(S) &\quad \text{conic hull} \\ \text{aff}(S) &\quad \text{affine hull} \\ \text{conv}(S) &\quad \text{convex hull} \end{aligned}$$





the convex hull of X

$$\textbf{conv}(X) = \{\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_n x_n \mid \mathbf{x}_i \in X, \lambda_1, \ldots, \lambda_n \geq 0 \text{ and } \sum_i \lambda_i = 1\}$$

Definitions

- rank of a matrix for columns (= for rows)
  if (m, n)-matrix has rank $= \min\{m, n\}$ then the matrix is full rank
  if (n, n)-matrix is full rank then it is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists\, \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

$$G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists\, \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} \leq \alpha\}$$

  ($\mathbf{a}^T\mathbf{x} = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subset \mathbb{R}^n$ is a polyhedron if $\exists\ m \in \mathbb{Z}^+, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$:

$$P = \{\mathbf{x} \in \mathbb{R} \mid A\mathbf{x} \leq \mathbf{b}\} = \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n \mid A_{i.}\mathbf{x} \leq b_i\}$$

- a polyhedron P is a polytope if it is bounded: $\exists\ B \in \mathbb{R}, B > 0$:

$$P \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq B\}$$

- Theorem: every polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

Definitions

- General optimization problem:
  $\max\{\varphi(\mathbf{x}) \mid \mathbf{x} \in F\}$, F is feasible region for $\mathbf{x}$
- Note: if F is open, eg, $x < 5$ then: $\sup\{x \mid x < 5\}$
  sumpreum: least element of $\mathbb{R}$ greater or equal than any element in F
- If A and $\mathbf{b}$ are made of rational numbers, $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ is a rational polyhedron

Definitions

- The inequality denoted by $(\mathbf{a}, \alpha)$ is called a valid inequality for P if $\mathbf{ax} \leq \alpha, \forall x \in P$.
  Note that $(\mathbf{a}, \alpha)$ is a valid inequality if and only if P lies in the half-space $\{x \in \mathbb{R}^n \mid \mathbf{ax} \leq \alpha\}$.
- A face of P is $F = \{\mathbf{x} \in P \mid \mathbf{ax} = \alpha\}$ where $(\mathbf{a}, \alpha)$ is a valid inequality for P. Hence, it is the intersection of P with the hyperplane of a valid inequality. It is said to be proper if $F \neq \emptyset$ and $F \neq P$.
- If $F \neq$ we say that it supports P.
- A point $\mathbf{x}$ for which $\{\mathbf{x}\}$ is a face is called a vertex of P and also a basic solution of $A\mathbf{x} \leq \mathbf{b}$ ($0\,\mathrm{dim}$ face)
- A facet is a maximal face distinct from P
  $\mathbf{cx} \leq \mathbf{d}$ is facet defining if $\mathbf{cx} = \mathbf{d}$ is a supporting hyperplane of P
  ($n - 1\,\mathrm{dim}$ face)

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$

Task:

1. decide that $\{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} \leq \mathbf{b}\}$ is empty (prob. infeasible), or
2. find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x}$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and $\exists$ solution
   1. one solution
   2. infinite solution
3. $F \neq \emptyset$ and $\nexists$ solution

Linear Programming and Linear Algebra

- Linear algebra: linear equations (Gaussian elimination)
- Integer linear algebra: linear diophantine equations
- Linear programming: linear inequalities (simplex method)
- Integer linear programming: linear diophantine inequalities

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Fundamental Theorem of LP

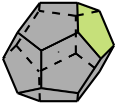
Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\} \;\; \textit{where } P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

If P is a bounded polyhedron and not empty and $\mathbf{x}^*$ is an optimal solution to the problem, then:

- *$\mathbf{x}^*$ is an extreme point (vertex) of P, or*
- *$\mathbf{x}^*$ lies on a face $F \subset P$ of optimal solution*



Proof idea:

- assume $\mathbf{x}^*$ not a vertex of P then $\exists$ a ball around it still in P. Show that a point in the ball has better cost
- if $\mathbf{x}^*$ is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

Implications:

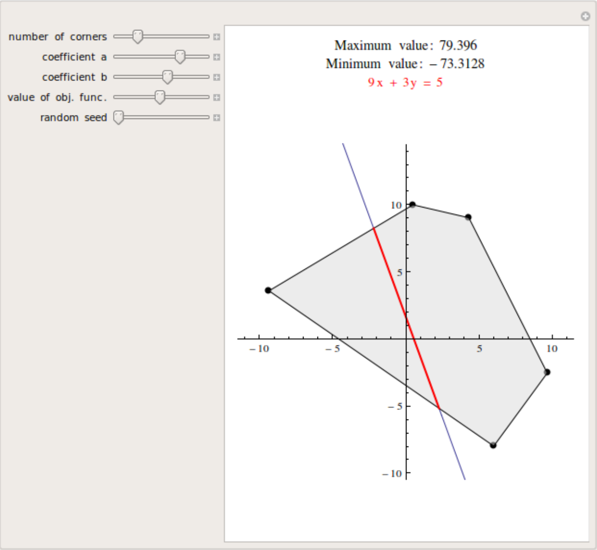
- the optimal solution is at the intersection of hyperplanes supporting halfspaces.
- hence finitely many possibilities
- Solution method: write all inequalities as equalities and solve all $\binom{n}{m}$ systems of linear equalities (n # variables, m # equality constraints)
- for each point we then need to check if feasible and if best in cost.
- each system is solved by Gaussian elimination
- Stirling approximation:

$$\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}} \text{ as } m \to \infty$$

Simplex Method

1. find a solution that is at the intersection of some n hyperplanes
2. try systematically to produce the other points by exchanging one hyperplane with another
3. check optimality, proof provided by duality theory

Demo


number of corners
coefficient a
coefficient b
value of obj. func.
random seed
Maximum value: 79.396
Minimum value: – 73.3128
9 x + 3 y = 5
10
5
– 5
– 10
– 10
– 5
5
10

Outline

1. Introduction
   Diet Problem

2. Solving LP Problems
   Fourier-Motzkin method

3. Preliminaries
   Fundamental Theorem of LP
   Gaussian Elimination

Gaussian Elimination

1. Forward elimination
   reduces the system to row echelon form by elementary row operations
   - multiply a row by a non-zero constant
   - interchange two rows
   - add a multiple of one row to anothe

   (or LU decomposition)

2. Back substitution (or reduced row echelon form - RREF)

Example

$$\begin{aligned} 2x + y - z &= 8 \quad (R1) \\ -3x - y + 2z &= -11 \quad (R2) \\ -2x + y + 2z &= -3 \quad (R3) \end{aligned}$$

R1	2	1	-1	8
R2	-3	-1	2	-11
R3	-2	1	2	-3

$$\begin{aligned} 2x + \phantom{\tfrac{1}{2}}y - \phantom{\tfrac{1}{2}}z &= 8 \quad (R1) \\ + \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 \quad (R2) \\ + 2y + 1z &= 5 \quad (R3) \end{aligned}$$

R1'=1/2 R1	1	1/2	-1/2	4
R2'=R2+3/2 R1	0	1/2	1/2	1
R3'=R3+R1	0	2	1	5

$$\begin{aligned} 2x + \phantom{\tfrac{1}{2}}y - \phantom{\tfrac{1}{2}}z &= 8 \quad (R1) \\ + \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 \quad (R2) \\ - \phantom{\tfrac{1}{2}}z &= 1 \quad (R3) \end{aligned}$$

R1'=R1	1	1/2	-1/2	4
R2'=2 R2	0	1	1	2
R3'=R3-4 R2	0	0	-1	1

$$\begin{aligned} 2x + \phantom{\tfrac{1}{2}}y - \phantom{\tfrac{1}{2}}z &= 8 \quad (R1) \\ + \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 \quad (R2) \\ - \phantom{\tfrac{1}{2}}z &= 1 \quad (R3) \end{aligned}$$

R1'=R1-1/2 R3	1	1/2	0	7/2
R2'=R2+R3	0	1	0	3
R3'=-R3	0	0	1	-1

$$\begin{aligned} x &= 2 \quad (R1) \\ y &= 3 \quad (R2) \\ z &= -1 \quad (R3) \end{aligned}$$

R1'=R1-1/2 R2	1	0	0	2	=> x=2
R2'=R2	0	1	0	3	=> y=3
R3'=R3	0	0	1	-1	=> z=-1

In Python


```
In [105]: import sympy as sy
          Ab=sy.Matrix([[ 2, 1, -1, 8],
                        [-3,-1,  2, -11],
                        [-2 ,  1,  2,  -3]])
          Ab.rref()

Out[105]: (Matrix([
          [1, 0, 0,  2],
          [0, 1, 0,  3],
          [0, 0, 1, -1]]), [0, 1, 2])
```

reduced row-echelon form of matrix and indices of pivot vars

LU Factorization

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = PLU$$

$$\mathbf{x} = A^{-1}\mathbf{b} = U^{-1}L^{-1}P^T\mathbf{b}$$

$$\mathbf{z}_1 = P^T\mathbf{b}, \quad \mathbf{z}_2 = L^{-1}\mathbf{z}_1, \quad \mathbf{x} = U^{-1}\mathbf{z}_2$$

```
In [117]: Ab[:,0:3].LUdecomposition()

Out[117]: (Matrix([
          [   1, 0, 0],
          [-3/2, 1, 0],
          [  -1, 4, 1]]),
           Matrix([
          [2,   1,   -1],
          [0, 1/2, 1/2],
          [0,   0,   -1]]),
           [])
```

Polynomial time $O(n^2m)$ but needs to guarantee that all the numbers during the run can be represented by polynomially bounded bits

Summary

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   - Gaussian Elimination