

DM545
Linear and Integer Programming

Lecture 3
The Simplex Method

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1. Simplex Method

- Standard Form

- Basic Feasible Solutions

- Algorithm

- Tableaux and Dictionaries

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A Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\max \quad [6 \ 8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

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Every LP problem can be converted in the **standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \mathbf{Ax} \leq & \mathbf{b} \\ \mathbf{x} \in & \mathbb{R}^n \end{aligned}$$

$$\mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

- if equations, then put two constraints, $ax \leq b$ and $ax \geq b$
- if $ax \geq b$ then $-ax \leq -b$
- if $\min \mathbf{c}^T \mathbf{x}$ then $\max(-\mathbf{c}^T \mathbf{x})$

and then be put in **equational standard form**

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \mathbf{Ax} = & \mathbf{b} \\ \mathbf{x} \geq & \mathbf{0} \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

1. “=” constraints
2. $\mathbf{x} \geq \mathbf{0}$ nonnegativity constraints
3. ($\mathbf{b} \geq \mathbf{0}$)
4. max

Transformation to Std Form

Every LP problem can be transformed in eq. std. form

1. introduce slack variables (or surplus)

$$5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 + x_4 = 40$$

2. if $x_1 \geq 0$ then $x_1 = x_1' - x_1''$
 $x_1' \geq 0$
 $x_1'' \geq 0$

3. ($b \geq 0$)

4. $\min c^T x \equiv \max(-c^T x)$

LP in $m \times n$ converted into LP with at most $(m + 2n)$ variables and m equations (n # original variables, m # constraints)

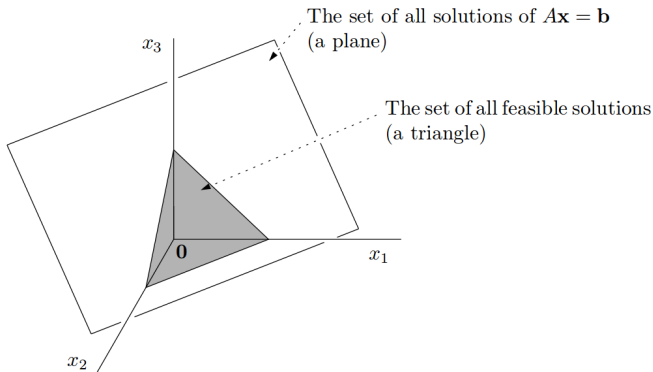
Geometry of LP in Eq. Std. Form

$$\max\{\mathbf{c}^T \mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

From linear algebra:

- the set of solutions of $\mathbf{Ax} = \mathbf{b}$ is an affine space (hyperplane not passing through the origin).
- $\mathbf{x} \geq \mathbf{0}$ nonnegative orthant (octant in \mathbb{R}^3)

In \mathbb{R}^3 :



- $Ax = b$ is a system of equations that we can solve by Gaussian elimination
- Elementary row operations of $[A \mid b]$ do not affect set of feasible solutions
 - multiplying all entries in some row of $[A \mid b]$ by a nonzero real number λ
 - replacing the i th row of $[A \mid b]$ by the sum of the i th row and j th row for some $i \neq j$
- We assume $n' \geq m$ and

$$\text{rank}([A \mid b]) = \text{rank}(A) = m$$

ie, rows of A are linearly independent
otherwise, remove linear dependent rows

1. Simplex Method

Standard Form

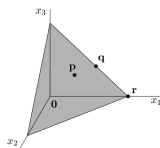
Basic Feasible Solutions

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Tableaux and Dictionaries

Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:

Let $B = \{1 \dots m\}$, $N = \{m + 1 \dots n + m\}$ be subsets partitioning the columns of A : A_B be made of columns of A indexed by B :

Definition

$\mathbf{x} \in \mathbb{R}^n$ is a **basic feasible solution** of the linear program $\max\{\mathbf{c}^T \mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ for an index set B if:

- $x_j = 0 \forall j \notin B$
- the square matrix A_B is nonsingular, ie, all columns indexed by B are lin. indep.
- $\mathbf{x}_B = A_B^{-1} \mathbf{b}$ is nonnegative, ie, $\mathbf{x}_B \geq \mathbf{0}$ (feasibility)

We call x_j for $j \in B$ **basic variables** and remaining variables **nonbasic variables**.

Theorem

A **basic feasible solution** is uniquely determined by the set B .

Proof:

$$Ax = A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$x_B = A_B^{-1} b$$

A_B is nonsingular hence one solution

Note: we call B a **(feasible) basis**

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

Theorem

Let P be a (convex) polyhedron from LP in std. form. For a point $v \in P$ the following are equivalent:

- (i) v is an extreme point (vertex) of P
- (ii) v is a basic feasible solution of LP

Proof: by recognizing that vertices of P are linear independent and such are the columns in A_B

Theorem

Let $LP = \max\{c^T x \mid Ax = b, x \geq 0\}$ be feasible and bounded, then the optimal solution is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming

Idea for solution method:

examine all basic solutions.

There are finitely many: $\binom{m+n}{m}$.

However, if $n = m$ then $\binom{2m}{m} \approx 4^m$.

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$$\max z = [6 \ 8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical eq. std. form: one decision variable is isolated in each constraint and does not appear in the other constraints nor in the obj. func. and b terms are positive

It gives immediately a basic feasible solution:

$$x_1 = 0, x_2 = 0, x_3 = 60, x_4 = 40$$

Is it optimal? Look at signs in $z \rightsquigarrow$ if positive then an increase would improve.

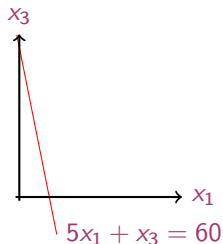
Let's try to increase a promising variable, ie, x_1 , one with positive coefficient in z

$$5x_1 + x_3 = 60$$

$$x_1 = \frac{60}{5} - \frac{x_3}{5}$$

$$x_3 = 60 - 5x_1 \geq 0$$

If $x_1 > 12$ then $x_3 < 0$

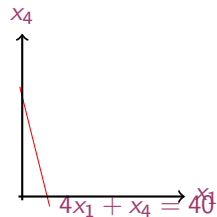


$$4x_1 + x_4 = 40$$

$$x_1 = \frac{40}{4} - \frac{x_4}{4}$$

$$x_4 = 40 - 4x_1 \geq 0$$

If $x_1 > 10$ then $x_4 < 0$



we can take the minimum of the two $\rightsquigarrow x_1$ increased to 10

x_4 exits the basis and x_1 enters

Simplex Tableau

First simplex tableau:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

we want to reach this new tableau

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	?	1	?	0	?
x_1	1	?	0	?	0	?
	0	?	0	?	1	?

Pivot operation:

1. Choose pivot:

column: one s with positive coefficient in obj. func.

row: ratio between coefficient b and pivot column: choose the one with smallest ratio:

$$\theta = \min_i \left\{ \frac{b_i}{a_{is}} : a_{is} > 0 \right\}, \quad \theta \text{ increase value of entering var.}$$

2. elementary row operations to update the tableau

- x_4 leaves the basis, x_1 enters the basis
 - Divide pivot row by pivot
 - Send to zero the coefficient in the pivot column of the first row
 - Send to zero the coefficient of the pivot column in the third (cost) row

	x_1	x_2	x_3	x_4	$-z$	b
I' = I - 5II'	0	5	1	-5/4	0	10
II' = II/4	1	1	0	1/4	0	10
III' = III - 6II'	0	2	0	-6/4	1	-60

From the last row we read: $2x_2 - 3/2x_4 - z = -60$, that is:

$$z = 60 + 2x_2 - 3/2x_4.$$

Since x_2 and x_4 are nonbasic we have $z = 60$ and $x_1 = 10, x_2 = 0, x_3 = 10, x_4 = 0$.

- Done? No! Let x_2 enter the basis

	x_1	x_2	x_3	x_4	$-z$	b
I' = I/5	0	1	1/5	-1/4	0	2
II' = II - I'	1	0	-1/5	1/2	0	8
III' = III - 2I'	0	0	-2/5	-1	1	-64

Definition (Reduced costs)

We call **reduced costs** the coefficients in the objective function of the nonbasic variables, \bar{c}_N

Proposition (Optimality Condition)

The basic feasible solution is **optimal** when the **reduced costs** in the corresponding simplex tableau are **nonpositive**, ie, such that:

$$\bar{c}_N \leq 0$$

Proof: Let z_0 be the obj value when $\bar{c}_N \leq 0$.

For any other feasible solution $\tilde{\mathbf{x}}$ we have:

$$\tilde{\mathbf{x}}_N \geq 0 \quad \text{and} \quad \mathbf{c}^T \tilde{\mathbf{x}} = z_0 + \bar{\mathbf{c}}_N^T \tilde{\mathbf{x}}_N \leq z_0$$

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$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} x_{n+i} &= b_i - \sum_{j=1}^n a_{ij} x_j, \quad i = 1, \dots, m \\ z &= \sum_{j=1}^n c_j x_j \end{aligned}$$

Tableau

$$\left[\begin{array}{c|c|c|c} I & \bar{A}_N & 0 & \bar{b} \\ \hline 0 & \bar{c}_N & 1 & -d \end{array} \right]$$

Dictionary

$$\begin{aligned} x_r &= \bar{b}_r - \sum_{s \notin B} \bar{a}_{rs} x_s, \quad r \in B \\ z &= \bar{d} + \sum_{s \notin B} \bar{c}_s x_s \end{aligned}$$

pivot operations in dictionary form:

choose col s with r.c. > 0

choose row with $\min\{-\bar{b}_i/\bar{a}_{is} \mid a_{is} < 0, i = 1, \dots, m\}$

update: express entering variable and substitute in other rows

Example

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

$$x_3 = 60 - 5x_1 - 10x_2$$

$$x_4 = 40 - 4x_1 - 4x_2$$

$$z = \quad + 6x_1 + 8x_2$$

After 2 iterations:

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	2
x_1	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

$$x_2 = 2 - 1/5x_3 + 1/4x_4$$

$$x_1 = 8 + 1/5x_3 - 1/2x_4$$

$$z = 64 - 2/5x_3 - 1x_4$$

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